이학박사 학위논문

Boundary behavior of harmonic functions for subordinate Brownian motion

(종속 브라운 운동에 대한 조화함수의 경계에서의 행동)

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Boundary behavior of harmonic functions for subordinate Brownian motion

A dissertation
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Abstract

In this thesis, we establish an oscillation estimate of nonnegative harmonic functions for a pure-jump subordinate Brownian motion. The infinitesimal generator of such subordinate Brownian motion X is an integro-differential operator. As an application, we give a probabilistic proof of the following form of relative Fatou theorem for such subordinate Brownian motion X in a bounded κ-fat open set; if u is a positive harmonic function with respect to X in a bounded κ-fat open set D and h is a positive harmonic function in D vanishing on D^c, then the non-tangential limit of u/h exists almost everywhere with respect to the Martin-representing measure of h. Under the gaugeability assumption, relative Fatou theorem is true for operators obtained from the generator of pure-jump subordinate Brownian motion in bounded κ-fat open set D through non-local Feynman-Kac transforms.

Key words: subordinate Brownian motion, relative Fatou type theorem, Martin kernel, Martin boundary, harmonic function, Martin representation
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Contents

Chapter 1

Introduction

Nowadays Lévy processes have been receiving intensive study due to their importance both in theories and applications. They are widely used in various fields, such as mathematical finance, actuarial mathematics and mathematical physics. Typically, the infinitesimal generators of general Lévy processes in $\mathbb{R}^d$ are not differential operators but integro-differential operators. Even though integro-differential operators are very important in the theory of partial differential equations, general Lévy processes and corresponding integro-differential operators are not easy to deal with. For a summary of some of these recent results from the probability literature, one can see [9] and the references therein. We refer readers to [12, 13] for samples of recent progresses in the PDE literature.

Let $W = (W_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ and $S = (S_t : t \geq 0)$ be a subordinator independent of W. The process $X = (X_t : t \geq 0)$ defined by $X_t = W_{S_t}$ is a rotationally invariant Lévy process in $\mathbb{R}^d$ and is called a subordinate Brownian motion. Subordinate Brownian motions form a very large class of Lévy processes. Nonetheless, compared with general Lévy processes, subordinate Brownian motions are much more tractable. If we take the Brownian motion W as given, then X is completely determined by the Laplace exponent of subordinator S. Hence one can deduce the properties of X from the subordinator S, or equivalently the Laplace exponent of it.

A smooth function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leq 0$ for every positive integer n. Every Bernstein function has a representation

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$, which is called the Lévy measure of ϕ. A Bernstein function ϕ is called complete if the Lévy measure μ of ϕ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \geq 0$ for every nonnegative integer n.

The purpose of this thesis is to give an oscillation estimate for (unbounded) harmonic functions (see Chapter 2 for the definition of harmonicity) for a large class of subordinate Brownian motions. Then using our estimates, we discuss non-tangential limits of the ratio of two harmonic functions for such subordinate Brownian motions.

Now we state the first main result of this thesis (see Theorem 3.2.9).

Theorem *Suppose that $X = (X_t : t \geq 0)$ is a Lévy process whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$ for $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \to [0, \infty)$ is a complete Bernstein function such that $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$, $\alpha \in (0, 2)$ and $\ell : (0, \infty) \to (0, \infty)$ is slowly varying at ∞. Then for every $\eta > 0$, there exists $a = a(\eta, \alpha, d, \ell) \in (0, 1)$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, 1]$,*

$$\sup_{x \in B(x_0, ar)} u(x) \leq (1 + \eta) \inf_{x \in B(x_0, ar)} u(x)$$

for every nonnegative function u in $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X.

Note that, for unlike a local operator, Theorem 3.2.9 cannot be obtained from the Harnack inequality and Moser's iteration method because harmonic functions in Theorem 3.2.9 are nonnegative in the whole space $\mathbb{R}^d$. On the other hand, if one just assumes that a harmonic function is nonnegative in $B(x_0, 2r)$, then even the Harnack inequality does not hold (see [24]).

Recently many results are obtained under the weaker assumption that ϕ is comparable to a regularly varying function at ∞ (see [26, 29, 30, 31]). But our technical Lemmas 3.1.2, 3.1.3 and 3.2.1 cannot be obtained under such assumptions.

Doob proved the relative Fatou theorem in the classical sense ([18]). That is, the ratio u/h of two positive harmonic functions with respect to Brownian motion on a unit open ball has non-tangential limits almost everywhere with respect to the Martin measure of h. Later, the relative Fatou theorem in the classical sense has been extended to some general open sets (see [38] and references therein). But the relative Fatou theorem stated above and the Fatou theorem are not true for harmonic functions for the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta^{\alpha/2})$ when $\alpha \in (0, 2)$ (see [5] for some counter examples). Correct formulation of the relative Fatou theorem for the integro-differential operator is the existence of non-tangential limits of the ratio u/h, where u is positive harmonic in an open set D and h is a positive harmonic function in D vanishing on D^c (see [10, 25, 27, 33]).

In this thesis, through a probabilistic method and Theorem 3.2.9, we show in Theorem 4.3.6 that the relative Fatou theorem holds for subordinate Brownian motion in very general open sets, namely, bounded κ-fat open sets, the family that includes bounded Lipschitz open sets. Further, under the gaugeability assumption, we show that relative Fatou theorem is also true under possibly discontinuous Feynman-Kac perturbation .

This thesis is organized as follows. In Chapter 2, we recall the definition of subordinate Brownian motion and its basic properties under our assumptions. In Chapter 3, we give the proof of Theorem 3.2.9. In these chapters, the influence of [11] in our results will be apparent. Chapter 4 contains the proof of the relative Fatou theorem in bounded κ-fat open sets. The main idea of our proof is similar to [25], which is inspired by Doob's approach (see also [1]). We use the Harnack and the boundary Harnack principle obtained in [28] and our Theorem 3.2.9. If the open set is the unit ball in $\mathbb{R}^2$, we show that our result is the best possible one. In Chapter 5, we recall the definition of Kato classes from [14, 25] and non-local Feynman-Kac transforms from [14, 15, 25]. Then under the gaugeability assumption, we show the relative Fatou theorem for non-local operators obtained from a bounded κ-fat open set through non-local Feynman-Kac transforms.

In the sequel, we will use the following convention: the value of the constant C_* will remain the same throughout this thesis, while the constants $c_0, c_1, c_2, \cdots$ signify constants whose values are unimportant and which may change from location to location. The labeling of the constants $c_0, c_1, c_2, \cdots$ starts anew in the statement of each result. We use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$ and $f(t) \sim g(t)$, $t \to 0$ ($f(t) \sim g(t)$, $t \to \infty$, respectively) means $\lim_{t\to 0} f(t)/g(t) = 1$ ($\lim_{t\to\infty} f(t)/g(t) = 1$, respectively). For any open set U, we denote $\delta_U(x) = \mathrm{dist}(x, U^c)$. Let $A(x, a, b) := \{y \in \mathbb{R}^d : a \le |x - y| < b\}$ and $B(x_0, r)$ be a ball in $\mathbb{R}^d$ centered at x_0 whose radius is r. When x_0 is the origin, we simply denote $B_r := B(0, r)$.

Chapter 2

Preliminaries

2.1 Subordinate Brownian motion

Suppose that $S = (S_t : t \geq 0)$ is a subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. A subordinator S is completely characterized by its Laplace exponent ϕ via

$$\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)) \quad \text{for } \lambda > 0.$$

A smooth function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leq 0$ for every positive integer n. Every Bernstein function has a representation

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$. a is called the killing coefficient, b is the drift and μ is the Lévy measure of the Bernstein function. Note that a nonnegative function ϕ on $(0, \infty)$ is the Laplace exponent of a subordinator if and only if it is a Bernstein function with $\phi(0+) = 0$. We also call μ the Lévy measure of the subordinator S. A Bernstein function ϕ is called a complete Bernstein function if μ has a completely monotone density $t \mapsto \mu(t)$, i.e., $\mu(t)dt = \mu(dt)$ and $(-1)^n D^n \mu \geq 0$ for every nonnegative integer n.

Let $W := (W_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ be a Brownian motion on $\mathbb{R}^d$ with

$\mathbb{P}_x(W_0 = x) = 1$ and $\mathbb{E}_x[e^{i\xi\cdot(W_t - W_0)}] = e^{-t|\xi|^2}$ for $\xi \in \mathbb{R}^d$, $t > 0$ and $x \in \mathbb{R}^d$. In the remainder of this thesis, we will use $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ to denote the subordinate Brownian motion defined by $X_t = W_{S_t}$, where $S = (S_t, t \geq 0)$ is a subordinator whose Laplace exponent is ϕ and S is independent of W.

Let

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt) \qquad \text{for } r > 0\,, \tag{2.1.1}$$

where μ is the Lévy measure of S. Then $J(x) := j(|x|)$ is the Lévy density of X. Note that the function $r \mapsto j(r)$ is strictly positive, continuous and decreasing on $(0, \infty)$.

Remark 2.1.1. *Since*

$$\left| \frac{\partial}{\partial r} \left(e^{-r^2/(4t)} \right) \right| = \left| \frac{4}{r} \frac{r^2}{8t} e^{-r^2/(8t)} \, e^{-r^2/(8t)} \right| \leq \frac{c}{r} e^{-r^2/(8t)}$$

for some constant $c > 0$ *and* $\int_0^\infty (4\pi t)^{-d/2} \, r^{-1} \, e^{-r^2/(8t)} \, \mu(t) dt \; = \; r^{-1} \, j(r/\sqrt{2})$, $j'(r)$ *is well-defined.*

For any open set $D \subset \mathbb{R}^d$, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t > 0 : \, X_t \notin D\}$. We define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemetery state.

It follows from [9, Chapter 5] that the process X has a transition density $p(t, x, y)$ which is jointly continuous. By the joint continuity and the strong Markov property, one can easily check that for $x, y \in D$,

$$p_D(t, x, y) \; := \; p(t, x, y) \; - \; \mathbb{E}_x[\, p(t - \tau_D, X_{\tau_D}, y)\, ; \, t > \tau_D]$$

is the transition density of X^D, which is jointly continuous (for example, see [26, Lemma 5.5]). For any bounded open set $D \subset \mathbb{R}^d$, we will use G_D to denote the Green function of X^D, i.e.,

$$G_D(x, y) := \int_0^\infty p_D(t, x, y) \, dt \qquad \text{for } x, y \in D.$$

Note that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$.

We define the Poisson kernel $P_D(x,y)$ as

$$P_D(x,y) := \int_D G_D(x,z)J(z-y)\,dz \qquad \text{for } (x,y) \in \mathbb{R}^d \times \overline{D}^c.$$

Thus we have for every bounded open subset D, function $f \geq 0$ and $x \in D$,

$$\mathbb{E}_x\left[f(X_{\tau_D});\ X_{\tau_D -} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} P_D(x,y)f(y)dy\,. \tag{2.1.2}$$

Note that, from the strong Markov property, it is well-known and easy to see that for every bounded open set $U \subset D$,

$$G_D(x,y) = G_U(x,y) + \mathbb{E}_x\left[G_D(X_{\tau_U},y)\right] \qquad \text{for } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.1.3}$$

Thus for every bounded open set $U \subset D$,

$$P_D(x,z) = P_U(x,z) + \mathbb{E}_x\left[P_D(X_{\tau_U},z)\right] \qquad \text{for } (x,z) \in U \times D^c. \tag{2.1.4}$$

2.2 Our hypothesis (A1) and its basic consequences

Throughout this thesis we will assume the following.
(A1) : *ϕ is a complete Bernstein function and regularly varying of index $\alpha/2$ at ∞ for some $\alpha \in (0,2)$. That is,*

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda) \tag{2.2.1}$$

for some $\alpha \in (0,2)$ and some positive function ℓ which is slowly varying at ∞.
Note that, this is an assumption about ϕ at ∞ and nothing is assumed about the behavior near zero. Clearly (2.2.1) implies that $b = 0$ and $\lambda \mapsto \ell(\lambda)$ is strictly positive and continuous on $(0,\infty)$. We refer the reader to [28] for examples. From [9, Proposition 5.23], we get

$$\mu(t) \sim \frac{\alpha}{2\Gamma(1-\alpha/2)} t^{-1}\phi(t^{-1}) \quad \text{as } t \to 0 \tag{2.2.2}$$

where $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$.

Recall $J(x) = j(|x|)$ is the Lévy density of X (see (2.1.1)). Applying [29, Lemma 13.3.1], we have the following.

Theorem 2.2.1.

$$j(r) \sim \frac{\alpha\Gamma((d+\alpha)/2)}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\alpha/2)}\frac{\phi(r^{-2})}{r^d} \qquad \textit{as } r \to 0.$$

As an immediate consequence of Theorem 2.2.1 and the continuity of $r \mapsto j(r)$ on $(0,\infty)$, we have the next corollary.

Corollary 2.2.2. *For every $R > 0$, there exists $c = c(R, \alpha, d, \ell) > 1$ such that for every positive y with $|y| \le R$,*

$$c^{-1}|y|^{-d}\phi(|y|^{-2}) \;\le\; J(y) \;\le\; c\,|y|^{-d}\phi(|y|^{-2}).$$

By [29, Proposition 13.3.5], the function $r \mapsto j(r)$ enjoys the following properties.

Proposition 2.2.3. *(1) For any $M > 0$, there exists $c_1 = c_1(M) > 0$ such that*

$$j(r) \le c_1 j(2r) \qquad \textit{for every } r \in (0, M).$$

(2) There exists $c_2 > 0$ such that

$$j(r) \le c_2 j(r+1) \qquad \textit{for every } r > 1.$$

We now recall the definition of harmonic functions with respect to X.

Definition 2.2.4. *Let D be an open subset in $\mathbb{R}^d$. A function u defined on $\mathbb{R}^d$ is said to be*

(1) *harmonic in D with respect to X if*

$$\mathbb{E}_x\left[|u(X_{\tau_B})|\right] < \infty \qquad \textit{and} \qquad u(x) = \mathbb{E}_x\left[u(X_{\tau_B})\right]$$

for every $x \in B$ and open set B whose closure is a compact subset of D;

(2) *regular harmonic in D with respect to X if it is harmonic in D with respect to X and for each $x \in D$,*

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) \right];$$

(3) *harmonic with respect to X^D if it is harmonic with respect to X in D and vanishes outside D.*

By [29, Corollary 13.4.8], we have the Harnack inequality.

Theorem 2.2.5. (Harnack inequality) *There exists a constant $C_0 > 0$ such that for every $r \in (0,1)$, $x_0 \in \mathbb{R}^d$ and function $f \geq 0$ in $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X, we have*

$$\sup_{y \in B(x_0, r/2)} f(y) \leq C_0 \inf_{y \in B(x_0, r/2)} f(y).$$

Using the continuities of G_D and J, one can easily check that P_D is continuous on $D \times \overline{D}^c$. Moreover, from [36, Theorem 1] we know that if V is a Lipschitz open set and $U \subset V$,

$$\mathbb{P}_x(X_{\tau_U} \in \partial V) = 0 \quad \text{ and } \quad \mathbb{P}_x(X_{\tau_U} \in dz) = P_U(x,z)\, dz \quad \text{on } V^c. \tag{2.2.3}$$

Thus every harmonic function u in D is written as

$$u(x) = \int_{B_r^c} P_{B_r}(x,y) u(y) dy \qquad \text{for } x \in B_r \subset \overline{B_r} \subset D\,. \tag{2.2.4}$$

When $r \leq 1$, by the continuity of $P_{B(x_0,r)}$ and the Harnack inequality (Theorem 2.2.5), we get

$$P_{B(x_0,r)}(x,y) \,\leq\, C_0\, P_{B(x_0,r)}(x_0,y) \quad \text{ for every } (x,y) \in B(x_0, r/2) \times \overline{B(x_0,r)}^c.$$

By the definition of the harmonicity, $P_{B(x_0,r)}(x_0,y)|u(y)| \in L^1(D)$ for $y \in \overline{B(x_0,r)}^c$. Thus we see that every harmonic function in D with respect to X is continuous by using Lebesgue dominated convergence theorem to (2.2.4).

The next two propositions are from [29, Propositions 13.4.10 and 13.4.13]. Recall that ϕ is the Laplace exponent of the subordinator S.

Proposition 2.2.6. *There exist $c_1, c_2 > 0$ such that for every $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$,*

$$P_{B(x_0,r)}(x,y) \le c_1\, j(|y-x_0|-r)\big(\phi(r^{-2})\phi((r-|x-x_0|)^{-2})\big)^{-1/2}$$

for $(x,y) \in B(x_0,r) \times \overline{B(x_0,r)}^{\,c}$, and

$$P_{B(x_0,r)}(x_0,y) \;\ge\; c_2\, \frac{j(|y-x_0|)}{\phi((r/2)^{-2})} \quad \text{for } \; y \in \overline{B(x_0,r)}^{\,c}.$$

Proposition 2.2.7. *For every $a \in (0,1)$, there exists $c = c(a) > 0$ such that for every $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$,*

$$P_{B(x_0,r)}(x,y) \;\le\; c\, r^{-d}\Big(\frac{\phi\big((|y-x_0|-r)^{-2}\big)}{\phi(r^{-2})}\Big)^{1/2}$$

for $x \in B(x_0, ar)$ and $y \in \{r < |x_0 - y| \le 2r\}$.

From [29, Lemmas 13.4.2 and 13.4.3], we have these following estimates on the mean exit times of balls.

Lemma 2.2.8. *(1) There exists a constant $c_1 = c_1(\alpha, d, \ell) > 0$ such that for every $r \in (0,1)$ and every $x \in \mathbb{R}^d$,*

$$\sup_{z \in B(x,r)} \mathbb{E}_z\left[\tau_{B(x,r)}\right] \le \frac{c_1}{\phi(r^{-2})}.$$

(2) For every $b \in (0,1)$, there exists a constant $c_2 = c_2(b, \alpha, d, \ell) > 0$ such that for every $r \in (0,1)$ and every $x \in \mathbb{R}^d$,

$$\inf_{z \in B(x,br)} \mathbb{E}_z\left[\tau_{B(x,r)}\right] \ge \frac{c_2}{\phi(r^{-2})}.$$

Chapter 3

Oscillation of harmonic functions

3.1 Estimates on Lévy density

Recall that S_t is a subordinator with Laplace exponent ϕ, W is a Brownian motion independent of S_t and $X_t = W_{S_t}$. First we show that ϕ being a complete Bernstein function implies that its Lévy density of X cannot decrease too fast in the following sense.

Lemma 3.1.1.

$$\lim_{\delta \downarrow 0} \sup_{t>1} \frac{\mu(t)}{\mu(t+\delta)} = 1.$$

Proof. Let $\eta > 0$ be given. Since μ is a completely monotone function, by Bernstein's theorem ([34, Theorem 1.4]) there exists a measure m on $[0,\infty)$ such that $\mu(t) = \int_{[0,\infty)} e^{-tx} m(dx)$. Choose $r = r(\eta) > 0$ such that

$$\eta \int_{[0,r]} e^{-x}\, m(dx) \geq \int_{(r,\infty)} e^{-x}\, m(dx).$$

Then for any $t > 1$, we have

$$\eta \int_{[0,r]} e^{-tx}\, m(dx) = \eta \int_{[0,r]} e^{-(t-1)x} e^{-x}\, m(dx) \geq e^{-(t-1)r} \eta \int_{[0,r]} e^{-x}\, m(dx)$$
$$\geq e^{-(t-1)r} \int_{(r,\infty)} e^{-x}\, m(dx) = \int_{(r,\infty)} e^{-(t-1)r} e^{-x}\, m(dx) \;\geq\; \int_{(r,\infty)} e^{-tx}\, m(dx).$$

Thus for any $t > 1$ and $\delta > 0$,

$$\begin{aligned}
\mu(t+\delta) &\geq \int_{[0,r]} e^{-(t+\delta)x}\, m(dx) \geq e^{-r\delta} \int_{[0,r]} e^{-tx}\, m(dx) \\
&= e^{-r\delta}(1+\eta)^{-1} \left(\int_{[0,r]} e^{-tx}\, m(dx) + \eta \int_{[0,r]} e^{-tx}\, m(dx) \right) \\
&\geq e^{-r\delta}(1+\eta)^{-1} \left(\int_{[0,r]} e^{-tx}\, m(dx) + \int_{(r,\infty)} e^{-tx}\, m(dx) \right) \\
&= e^{-r\delta}(1+\eta)^{-1} \int_{[0,\infty)} e^{-tx}\, m(dx) = \; e^{-r\delta}(1+\eta)^{-1} \mu(t).
\end{aligned}$$

Therefore,

$$\limsup_{\delta \downarrow 0} \left(\sup_{t>1} \frac{\mu(t)}{\mu(t+\delta)} \right) \;\leq\; 1+\eta\,.$$

Since $\eta > 0$ is arbitrary and $\frac{\mu(t)}{\mu(t+\delta)} \geq 1$, we conclude that this lemma holds.

□

Lemma 3.1.2.

$$\limsup_{\delta \downarrow 0} \sup_{r>2} \frac{j(r)}{j(r+\delta)} = 1.$$

Proof. Fix $\varepsilon \in (0,1)$ and let $L := \dfrac{\alpha}{2\Gamma(1-\alpha/2)}$. Using (2.2.1), (2.2.2) and the fact that ℓ is slowly varying, we choose $t_* = t_*(\varepsilon) \in (0, 1/2)$ such that for every $t \leq 2\, t_*$,

$$(1+\varepsilon)^{-1} L \frac{\phi(t^{-1})}{t} \;\leq\; \mu(t) \leq (1+\varepsilon)\, L \frac{\phi(t^{-1})}{t}$$

and

$$1 \le \frac{\phi\big((1+\varepsilon)t^{-1}\big)}{\phi(t^{-1})} \le (1+\varepsilon)^{1+\alpha/2}.$$

Then we get

$$\begin{aligned} \mu\big((1+\varepsilon)t\big) &\ge (1+\varepsilon)^{-1} L \frac{\phi\big((1+\varepsilon)^{-1}t^{-1}\big)}{(1+\varepsilon)t} \ge (1+\varepsilon)^{-3-\alpha/2} L \frac{\phi(t^{-1})}{t} \\ &\ge (1+\varepsilon)^{-4-\alpha/2}\mu(t) \quad \text{for every } t \le 2\,t_*. \end{aligned} \tag{3.1.1}$$

Now using Lemma 3.1.1, we choose $\delta_1 \in (0, \varepsilon(1+\varepsilon)^{-1}]$ such that for every $t \ge 1$,

$$\mu(t+\delta_1) \le \mu(t) \le (1+\varepsilon)\mu(t+\delta_1). \tag{3.1.2}$$

Since

$$\frac{\mu(t) - \mu\big((1-\delta)^{-1}t\big)}{\mu\big((1-\delta)^{-1}t\big)} \le \frac{\mu(t) - \mu\big((1-\delta)^{-1}t\big)}{\mu(4)}$$

and

$$\frac{\mu(t) - \mu(\delta+t)}{\mu(\delta+t)} \le \frac{\mu(t) - \mu(\delta+t)}{\mu(4)}$$

for every $\delta \in (0, 1/2)$ and $t \in [t_*, 2]$, by using the continuity of μ, we choose $\delta_2 \in (0, \delta_1]$ such that for every $t \in [t_*, 2]$,

$$\mu(t) \le (1+\varepsilon)\,\mu\big(t(1-\delta_2)^{-1}\big) \quad \text{and} \quad \mu(t) \le (1+\varepsilon)\,\mu(t+\delta_2)\,. \tag{3.1.3}$$

Combining (3.1.1)–(3.1.3), we have that for every $\delta \le \delta_2$,

$$\mu(t) \le (1+\varepsilon)^{4+\alpha/2} \times \begin{cases} \mu\big(t(1-\delta)^{-1}\big) & \text{when } t < 2 \\ \mu(t+\delta) & \text{when } t \ge 1/2. \end{cases} \tag{3.1.4}$$

Let $r > 2$. Using (2.1.1), we put

$$j(r+\delta) = \left(\int_0^1 + \int_1^\infty\right) (4\pi t)^{-d/2} \exp\Big(-\frac{(r+\delta)^2}{4t}\Big)\,\mu(t)\,dt =: I + II.$$

Since $(1-\delta)(r+\delta)^2 \le r^2 + \delta(r+\delta)\big(2-(r+\delta)\big) \le r^2$, by (3.1.4) and a change of variables,

$$\begin{aligned}
I &\geq \int_0^1 (4\pi t)^{-d/2} \exp\big(-\frac{(1-\delta)^{-1} r^2}{4t}\big)\, \mu(t)\, dt \\
&= (1-\delta)^{-1+d/2} \int_0^{1-\delta} (4\pi t)^{-d/2} \exp(-\frac{r^2}{4t})\, \mu\big(t(1-\delta)^{-1}\big)\, dt \\
&\geq (1-\delta)^{-1+d/2} (1+\varepsilon)^{-4-\alpha/2} \int_0^{1-\delta} (4\pi t)^{-d/2} \exp(-\frac{r^2}{4t})\, \mu(t)\, dt
\end{aligned}$$

for every $\delta \leq \delta_2$. On the other hand, from $0 \leq (r+\delta-t)^2 = (r+\delta)^2 - 2tr + t(t-\delta) - \delta t$, we see that $t(t-\delta) \geq 2tr + \delta t - (r+\delta)^2$. Thus we get

$$\frac{(r+\delta)^2}{4t} - \frac{r^2}{4(t-\delta)} = \frac{(r+\delta)^2(t-\delta) - r^2 t}{4t(t-\delta)} = \frac{\delta(2tr + \delta t - (r+\delta)^2)}{4t(t-\delta)} \leq \frac{\delta}{4}.$$

Therefore by using this, a change of variables, (3.1.4) and the inequality $t + \delta \leq t\,(1-\delta)^{-1}$ for $1-\delta \leq t < \infty$, we get

$$\begin{aligned}
II &\geq e^{-\delta/4} \int_1^\infty (4\pi t)^{-d/2} \exp\big(-\frac{r^2}{4(t-\delta)}\big)\, \mu(t)\, dt \\
&= e^{-\delta/4} \int_{1-\delta}^\infty (4\pi(t+\delta))^{-d/2} \exp(-\frac{r^2}{4t})\, \mu(t+\delta)\, dt \\
&\geq e^{-\delta/4} (1+\varepsilon)^{-4-\alpha/2} (1-\delta)^{d/2} \int_{1-\delta}^\infty (4\pi t)^{-d/2} \exp(-\frac{r^2}{4t})\, \mu(t)\, dt
\end{aligned}$$

for every $\delta \leq \delta_2$. Consequently for every $\delta \leq \delta_2$ and $r > 2$,

$$j(r+\delta) \geq \big((1-\delta)^{-1+d/2} \wedge e^{-\delta/4}(1-\delta)^{d/2}\big)(1+\varepsilon)^{-4-\alpha/2} j(r)$$

and so

$$\limsup_{\delta \downarrow 0} \left(\sup_{r>2} \frac{j(r)}{j(r+\delta)} \right) \leq (1+\varepsilon)^{4+\alpha/2}.$$

Since $\varepsilon > 0$ is arbitrary and $\dfrac{j(r)}{j(r+\delta)} \geq 1$, the proof is completed. $\square$

Lemma 3.1.3.

$$\lim_{\delta\downarrow 0} \sup_{r\in(0,4]} \frac{j(r)}{j\big(r(1+\delta)\big)} = 1\,.$$

Proof. Fix $\varepsilon > 0$ and let $\mathcal{A} := \alpha\Gamma((d+\alpha)/2)2^{-1+\alpha}\pi^{-d/2}(\Gamma(1-\alpha/2))^{-1}$. By Potter's Theorem [7, Theorem 1.5.6(i)], there exists $r_1 = r_1(\varepsilon) > 0$ such that

$$\frac{\ell(t^{-2})}{\ell(s^{-2})} \;\geq\; (1+\varepsilon)^{-1} \min\left\{\frac{t}{s}\,,\,\frac{s}{t}\right\} \qquad \text{for } s,t \leq 2\,r_1.$$

Moreover by Theorem 2.2.1, there exists $r_2 = r_2(\varepsilon) > 0$ such that

$$1+\varepsilon \;\geq\; \frac{\mathcal{A}\ell(s^{-2})}{s^{d+\alpha}j(s)} \;\geq\; (1+\varepsilon)^{-1} \quad \text{for } s \leq 2\,r_2.$$

Thus for $r \leq r_3 := r_1 \wedge r_2$ and $\delta \in (0,1)$,

$$\begin{aligned}
\frac{j\big(r(1+\delta)\big)}{j(r)} \quad &= \quad \left(\frac{j\big(r(1+\delta)\big)\, r^{d+\alpha}\,(1+\delta)^{d+\alpha}}{\mathcal{A}\,\ell\big(r^{-2}(1+\delta)^{-2}\big)}\right)\left(\frac{\mathcal{A}\,\ell(r^{-2})}{r^{d+\alpha}j(r)}\right)\\
&\qquad \times \frac{\ell\big(r^{-2}(1+\delta)^{-2}\big)}{\ell(r^{-2})}\,(1+\delta)^{-d-\alpha}\\
&\geq \quad (1+\varepsilon)^{-3}(1+\delta)^{-d-\alpha-1}\,.
\end{aligned}$$

On the other hand for every $\delta \in (0,1)$ and $r \in [r_3, 4]$,

$$\begin{aligned}
\frac{j(r) - j\big((1+\delta)r\big)}{j\big((1+\delta)r\big)} \quad &\leq \quad \frac{j(r) - j\big((1+\delta)r\big)}{j(8)} \;\leq\; j(8)^{-1}\delta r|j\,'(r_3)|\\
&\leq \quad 4j(8)^{-1}\delta|j\,'(r_3)|
\end{aligned}$$

and so $\big(1 + 4j(8)^{-1}\delta|j\,'(r_3)|\big)\, j(r(1+\delta)) \;\geq\; j(r)$. Therefore

$$\limsup_{\delta\downarrow 0}\left(\sup_{r\in(0,4]} \frac{j(r)}{j\big(r(1+\delta)\big)}\right) \leq (1+\varepsilon)^3.$$

Since $\varepsilon > 0$ is arbitrary and $\dfrac{j(r)}{j\big(r(1+\delta)\big)} \geq 1$, we complete the proof. $\square$

3.2 Oscillation

In this section, for the notational convention we define

$$\Lambda_{a,b}(u) := \int_{A(0,a,b)} j(|y|)u(y)dy \quad \text{and} \quad \Lambda_a(u) := \int_{B_a^c} j(|y|)u(y)dy$$

for every nonnegative function u on $\mathbb{R}^d$ and constants a, b with $b > a > 0$. By Lemmas 3.1.2 and 3.1.3, there exists an increasing continuous function $\delta(\varepsilon) : (0, 1/2] \to (0, 1/2]$ such that $\lim_{\varepsilon\downarrow 0} \delta(\varepsilon) = 0$ and

$$\left(\sup_{r>2} \frac{j(r)}{j(r+\delta(\varepsilon))}\right) \vee \left(\sup_{r\in(0,4]} \frac{j(r)}{j(r(1+\delta(\varepsilon)))}\right) \leq 1+\varepsilon. \tag{3.2.1}$$

Lemma 3.2.1. *For every $0 < \varepsilon \leq 1/2$, $0 < p \leq 1/2$, $r \leq 2$ and any nonnegative function u in $\mathbb{R}^d$, we have for every $x \in B_{\delta pr/3}$*

$$(1+\varepsilon)^{-1}\Lambda_{pr}(u)\mathbb{E}_x[\tau_{B_{\delta pr/3}}] \leq \int_{B_{pr}^c} P_{B_{\delta pr/3}}(x,y)u(y)dy \leq (1+\varepsilon)\Lambda_{pr}(u)\mathbb{E}_x[\tau_{B_{\delta pr/3}}]$$

where $\delta = \delta(\varepsilon) \in (0, 1/2]$ is in (3.2.1).

Proof. If $z \in B_{\delta pr/3}$ and $y \in A(0, pr, 1)$, then we have

$$|y-z| \leq |y| + |z| \leq |y| + \delta pr/3 \leq (1+\delta/3)|y| \leq (1+\delta)|y|$$

and

$$|y-z| \geq |y| - |z| \geq |y| - \delta pr/3 \geq (1-\delta/3)|y| \geq (1+\delta)^{-1}|y|.$$

Thus by (3.2.1) and the fact that $r \mapsto j(r)$ is decreasing,

$$1+\varepsilon \geq \frac{j((1+\delta)^{-1}|y|)}{j(|y|)} \geq \frac{j(|y-z|)}{j(|y|)} \geq \frac{j((1+\delta)|y|)}{j(|y|)} \geq (1+\varepsilon)^{-1}$$

for $y \in A(0, pr, 1)$. On the other hand, since the assumptions $r \leq 2$ and

$p \le 1/2$ imply $\delta pr/3 \le \delta$, we have

$$|y-z| \le |y| + |z| \le |y| + \delta pr/3 \le |y| + \delta$$

and

$$|y-z| \ge |y| - |z| \ge |y| - \delta pr/3 \ge |y| - \delta.$$

Thus by (3.2.1) and the fact that j is decreasing,

$$1+\varepsilon \;\ge\; \frac{j(|y|-\delta)}{j(|y|)} \;\ge\; \frac{j(|y-z|)}{j(|y|)} \;\ge\; \frac{j(|y|+\delta)}{j(|y|)} \;\ge\; (1+\varepsilon)^{-1} \quad \text{for } |y| \ge 1\,.$$

So we have for $x \in B_{\delta pr/3}$,

$$\int_{B_{pr}^c} P_{B_{\delta pr/3}}(x,y)u(y)dy \;=\; \int_{B_{pr}^c}\int_{B_{\delta pr/3}} G_{B_{\delta pr/3}}(x,z)j(|z-y|)dz\,u(y)dy$$
$$\le\; (1+\varepsilon)\int_{B_{\delta pr/3}} G_{B_{\delta pr/3}}(x,z)dz \int_{B_{pr}^c} j(|y|)u(y)dy \;=\; (1+\varepsilon)\mathbb{E}_x[\tau_{B_{\delta pr/3}}]\Lambda_{pr}(u)$$

and

$$\begin{aligned}\int_{B_{pr}^c} P_{B_{\delta pr/3}}(x,y)u(y)dy \;&\ge\; (1+\varepsilon)^{-1}\int_{B_{\delta pr/3}} G_{B_{\delta pr/3}}(x,z)dz \int_{B_{pr}^c} j(|y|)u(y)dy \\ &=\; (1+\varepsilon)^{-1}\mathbb{E}_x[\tau_{B_{\delta pr/3}}]\Lambda_{pr}(u).\end{aligned}$$

□

The next inequalities will be used several times in the remainder of this thesis.

Lemma 3.2.2. *There exists $c = c(\alpha, d, \ell) > 0$ such that*

$$s^d\phi(s^{-2}) \le c\, r^d\phi(r^{-2}) \qquad \textit{for } 0 < s < r \le 4, \tag{3.2.2}$$

$$s\left(\phi(s^{-2})\right)^{1/2} \le c\, r\left(\phi(r^{-2})\right)^{1/2} \qquad \textit{for } 0 < s < r \le 4, \tag{3.2.3}$$

$$\int_0^r \left(\phi(s^{-2})\right)^{1/2} ds \le c\, r\left(\phi(r^{-2})\right)^{1/2} \qquad \textit{for } 0 < r \le 4, \tag{3.2.4}$$

$$\int_r^\infty \frac{\phi(s^{-2})}{s} ds \le c\,\phi(r^{-2}) \qquad \textit{for } 0 < r \le 4 \tag{3.2.5}$$

and

$$\int_0^r \frac{1}{s\,\phi(s^{-2})} ds \le c\,\frac{1}{\phi(r^{-2})} \qquad \textit{for } 0 < r \le 4. \tag{3.2.6}$$

Proof. The first two inequalities follow easily from (2.2.1) and [7, Theorem 1.5.3], while the last three from (2.2.1) and the 0-version of [7, Theorem 1.5.11]. □

Lemma 3.2.3. *For every $p \in (0, 1)$, there exists $c = c(\alpha, d, \ell, p) > 0$ such that for every $r \in (0, 1]$ and $(x, y) \in B_{pr} \times A(0, r(1 + p)/2, r)$,*

$$\int_{r(1+p)/2}^{|y|} P_{B_s}(x, y)\, ds \le \frac{c\, r}{\phi(r^{-2})}\, j(|y|)\,.$$

Proof. Let $0 < p < 1$ and $q = (1 + p)/2$. From Proposition 2.2.7, we get for $x \in B(0, pr)$ and $y \in A(0, qr, r)$,

$$\int_{qr}^{|y|} P_{B_s}(x, y) ds \le c_1 \int_{qr}^{|y|} s^{-d}\Big(\frac{\phi\big((|y| - s)^{-2}\big)}{\phi(s^{-2})}\Big)^{1/2} ds\,.$$

Note that by (3.2.3),

$$\int_{qr}^{|y|} s^{-d}\Big(\frac{\phi\big((|y| - s)^{-2}\big)}{\phi(s^{-2})}\Big)^{1/2} ds \le c_2\, \frac{(qr)^{-d}}{\big(\phi\big((qr)^{-2}\big)\big)^{1/2}} \int_{qr}^{|y|} \big(\phi\big((|y|-s)^{-2}\big)\big)^{1/2} ds\,.$$

Moreover by (3.2.4),

$$\begin{aligned}\int_{qr}^{|y|} \left(\phi\big((|y|-s)^{-2}\big)\right)^{1/2} ds &= \int_0^{|y|-qr} \left(\phi(s^{-2})\right)^{1/2} ds \\ &\le c_3\,(|y|-qr)\left(\phi\big((|y|-qr)^{-2}\big)\right)^{1/2}.\end{aligned}$$

Using (3.2.3) again, we get

$$(|y|-qr)\big(\phi\big((|y|-qr)^{-2}\big)\big)^{1/2} \le c_4\,|y|\left(\phi(|y|^{-2})\right)^{1/2}$$

for some constant $c_4 > 0$. Thus by using the fact $(qr)^{-d} \le 2^d\,|y|^{-d}$, we have

$$\begin{aligned}\int_{qr}^{|y|} P_{B_s}(x,y)ds &\le c_1c_2c_3c_4 \frac{|y|\big(\phi(|y|^{-2})\big)^{1/2}}{(qr)^d\big(\phi\big((qr)^{-2}\big)\big)^{1/2}} \\ &\le c_1c_2c_3c_4 \frac{2^d\,|y|^{-d+1}\,\phi(|y|^{-2})}{\big(\phi\big((qr)^{-2}\big)\big)^{1/2}\big(\phi(|y|^{-2})\big)^{1/2}} \\ &\le c_5 \frac{|y|^{-d+1}\phi(|y|^{-2})}{\phi(r^{-2})}\end{aligned}$$

for some constant $c_5 > 0$. The last inequality comes from the increasing property of ϕ. By using Corollary 2.2.2, this lemma is proved. □

Lemma 3.2.4. *For every $p \in (0,1)$, there exists $c = c(\alpha, d, \ell, p) > 0$ such that for every $r \in (0,1)$ and $(x,y) \in B_{pr} \times B_r^c$,*

$$P_{B_r}(x,y) \le \frac{c}{\phi(r^{-2})}\left(\int_{A(0,(1+p)r/2,r)} j(|z|)P_{B_r}(z,y)dz + j(|y|)\right).$$

Proof. Fix $0 < p < 1$ and define $q_1 := (1+p)/2$ and $q_2 := (3+p)/4$. By (2.1.4) and (2.2.3), we have

$$\begin{aligned}P_{B_r}(x,y) &= \mathbb{E}_x[P_{B_r}(X_{\tau_{B_s}}, y)] + P_{B_s}(x,y) \\ &= \int_{B_r\setminus B_s} P_{B_r}(z,y)P_{B_s}(x,z)dz + P_{B_s}(x,y)\end{aligned}$$

for every $s \in [q_1 r, q_2 r]$ and $x \in B_{pr}$. Thus

$$\begin{aligned} P_{B_r}(x,y) &= \frac{1}{r(q_2-q_1)} \int_{q_1 r}^{q_2 r} \int_{B_r \setminus B_s} P_{B_r}(z,y) P_{B_s}(x,z) dz ds \\ &\quad + \frac{1}{r(q_2-q_1)} \int_{q_1 r}^{q_2 r} P_{B_s}(x,y) ds \\ &=: I + II. \end{aligned}$$

By Tonelli theorem, we have

$$\begin{aligned} I &= \frac{4}{r(1-p)} \int_{q_1 r}^{q_2 r} \int_{\{z \in B_r; |z| \geq q_1 r\}} 1_{\{|z| \geq s\}} P_{B_s}(x,z) P_{B_r}(z,y) dz ds \\ &\leq \frac{4}{r(1-p)} \int_{B_r \setminus B_{q_1 r}} \left(\int_{q_1 r}^{|z|} P_{B_s}(x,z) ds \right) P_{B_r}(z,y) dz. \end{aligned}$$

Apply Lemma 3.2.3 to the last inequality above, we get

$$I \leq c_1 \int_{B_r \setminus B_{q_1 r}} \frac{j(|z|)}{\phi(r^{-2})} P_{B_r}(z,y) dz. \tag{3.2.7}$$

On the other hand, by Proposition 2.2.6,

$$II \leq c_2 \frac{1}{r(q_2-q_1)} \int_{q_1 r}^{q_2 r} \frac{j(|y|-s)}{\big(\phi(s^{-2})\, \phi((s-|x|)^{-2})\big)^{1/2}} ds.$$

Note that for every $s \in [q_1 r, q_2 r]$, we have $(1-q_2)|y| \leq |y|-s$ for $y \in A(0,r,4)$, while when $|y| \geq 4$ we have $|y|-s \geq |y|-1$. Since $s-|x| \leq s \leq q_2 r < r$, we have by increasing property of ϕ and the monotonicity of j,

$$\frac{j(|y|-s)}{\big(\phi(s^{-2})\, \phi((s-|x|)^{-2})\big)^{1/2}} \leq \frac{c_3\, j\big((1-q_2)|y|\big)}{\phi(r^{-2})} \quad \text{for } y \in A(0,r,4)$$

and

$$\frac{j(|y|-s)}{\big(\phi(s^{-2})\, \phi((s-|x|)^{-2})\big)^{1/2}} \leq \frac{c_4\, j(|y|-1)}{\phi(r^{-2})} \quad \text{for } |y| \geq 4\,.$$

By applying Proposition 2.2.3 inductively, we get

$$\frac{j(|y|-s)}{\big(\phi(s^{-2})\,\phi((s-|x|)^{-2})\big)^{1/2}} \leq \frac{c_5\, j(|y|)}{\phi(r^{-2})}$$

for some constant $c_5 > 0$. Therefore

$$II \leq \frac{c_6\, j(|y|)}{\phi(r^{-2})}. \tag{3.2.8}$$

Combining (3.2.7) and (3.2.8), we conclude that

$$P_{B_r}(x,y) \leq \frac{c_7}{\phi(r^{-2})} \left(\int_{B_r \setminus B_{q_1 r}} j(|z|) P_{B_r}(z,y) dz + j(|y|) \right).$$

$\square$

Note that since ℓ is slowly varying at ∞ and ℓ is strictly positive and continuous on $(0,\infty)$, there exists a constant $c = c(\alpha,\ell) > 1$ such that for every $r \in (0,1)$,

$$c^{-1} \leq \frac{\ell\big((2r/3)^{-2}\big)}{\ell(r^{-2})} \leq \left(\frac{\ell\big((2r/3)^{-2}\big)}{\ell(r^{-2})} \vee \frac{\ell\big((r/2)^{-2}\big)}{\ell(r^{-2})} \right) \leq c. \tag{3.2.9}$$

Lemma 3.2.5. *There exists $c = c(\alpha,d,\ell) > 1$ such that for every $r \in (0,1)$ and $(x,y) \in B_{r/2} \times B_r^c$,*

$$P_{B_r}(x,y) \geq \frac{c}{\phi(r^{-2})} \left(\int_{A(0,r/2,r)} j(|z|) P_{B_r}(z,y) dz + j(|y|) \right).$$

Proof. Fix $r \in (0,1)$. By (2.1.4) and (2.2.3),

$$\begin{aligned}
P_{B_r}(x,y) &= \mathbb{E}_x[P_{B_r}(X_{\tau_{B_{2r/3}}}, y)] + P_{B_{2r/3}}(x,y) \\
&= \int_{B_{3r/4}\setminus B_{2r/3}} P_{B_r}(z,y)\mathbb{P}_x(X_{\tau_{B_{2r/3}}} \in dz) \\
&\quad + \int_{B_r\setminus B_{3r/4}} P_{B_r}(z,y)P_{B_{2r/3}}(x,z)dz + P_{B_{2r/3}}(x,y) \\
&\geq \int_{B_r\setminus B_{3r/4}} P_{B_r}(z,y)\int_{B_{2r/3}} G_{B_{2r/3}}(x,w)j(|z-w|)dwdz \\
&\quad + \int_{B_{2r/3}} G_{B_{2r/3}}(x,w)j(|y-w|)dw \\
&=: I + II. \qquad (3.2.10)
\end{aligned}$$

Note that if $4 > |z| \geq 3r/4$ and $|w| < 2r/3$, then $|z-w| \leq |z| + |w| \leq |z| + 2r/3 \leq 2|z|$. Since j is decreasing, by Proposition 2.2.3 (1), $c_1^{-1}j(|z|) \leq j(2|z|) \leq j(|z-w|)$ for some constant $c_1 > 0$. If $4 \leq |z|$ and $|w| < 2r/3$, then $|z-w| \leq |z| + 2r/3$ and by Proposition2.2.3 (2),

$$c_2^{-1}j(|z|) \leq c_2^{-1}j(|z| + 2r/3 - 1) \leq j(|z| + 2r/3) \leq j(|z-w|)$$

for some constant $c_2 > 0$. Thus using (2.2.1), Lemma 2.2.8 (2) and (3.2.9), we have

$$\begin{aligned}
I &\geq c_3\,\mathbb{E}_x[\tau_{B_{2r/3}}]\int_{B_r\setminus B_{3r/4}} j(|z|)P_{B_r}(z,y)dz \\
&\geq \frac{c_4}{\phi(r^{-2})}\int_{B_r\setminus B_{3r/4}} j(|z|)P_{B_r}(z,y)dz \qquad (3.2.11)
\end{aligned}$$

and

$$II \geq c_5\,\mathbb{E}_x[\tau_{B_{2r/3}}]\,j(|y|) \geq \frac{c_6}{\phi(r^{-2})}j(|y|)\,. \qquad (3.2.12)$$

By Corollary 2.2.2 and Lemma 3.2.4,

$$\begin{aligned}
&\int_{B_{3r/4}\setminus B_{r/2}} j(|z|)P_{B_r}(z,y)dz \le \Big(\sup_{z\in B_{3r/4}} P_{B_r}(z,y)\Big)\int_{A(0,r/2,3r/4)} j(|y|)dy, \\
&\le \frac{c_7}{\phi(r^{-2})}\int_{r/2}^{3r/4} s^{-1}\phi(s^{-2})ds\Big(\int_{B_r\setminus B_{3r/4}} j(|z|)P_{B_r}(z,y)dz + j(|y|)\Big) \\
&\le \frac{c_7}{\phi(r^{-2})}\int_{r/2}^{\infty} s^{-1}\phi(s^{-2})ds\Big(\int_{B_r\setminus B_{3r/4}} j(|z|)P_{B_r}(z,y)dz + j(|y|)\Big).
\end{aligned}$$

Applying (2.2.1), (3.2.5) and (3.2.9), we obtain

$$\begin{aligned}
&\int_{B_{3r/4}\setminus B_{r/2}} j(|z|)\,P_{B_r}(z,y)dz \\
\le\ & c_8\, 2^\alpha\, \frac{\ell\big((r/2)^{-2}\big)}{\ell(r^{-2})}\Big(\int_{B_r\setminus B_{3r/4}} j(|z|)\,P_{B_r}(z,y)dz + j(|y|)\Big) \\
\le\ & c_9\Big(\int_{B_r\setminus B_{3r/4}} j(|z|)\,P_{B_r}(z,y)dz + j(|y|)\Big). \qquad (3.2.13)
\end{aligned}$$

Combining (3.2.10)–(3.2.13), we have proved the lemma. □

Recall that C_0 is the constant in Theorem 2.2.5.

Lemma 3.2.6. *There exists $C_* = C_*(\alpha, d, \ell) \ge C_0$ such that for every $r \in (0,1)$, any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in B_r with respect to X and for any $x \in B_{r/2}$,*

$$\begin{aligned}
C_*^{-1}\mathbb{E}_x[\tau_{B_r}]\Lambda_{r/2}(u) \le u(x) &\le C_*\,\mathbb{E}_x[\tau_{B_{2r/3}}]\Lambda_{3r/4}(u) \qquad (3.2.14)\\
&\le C_*\,\mathbb{E}_x[\tau_{B_r}]\Lambda_{r/2}(u). \qquad (3.2.15)
\end{aligned}$$

Proof. Since u is regular harmonic in B_r with respect to X, by (2.2.3) we have $u(x) = \int_{B_r^c} P_{B_r}(x,y)\,u(y)\,dy$ for every $x \in B_r$. Thus by using Lemma

3.2.4 in the first inequality and (3.2.9) in the second, we get

$$\begin{aligned}
u(x) &\leq \frac{c_1}{\phi(r^{-2})}\Big(\int_{B_r^c}\int_{A(0,3r/4,r)} j(|z|)P_{B_r}(z,y)dz\, u(y)\, dy + \int_{B_r^c} j(|y|)u(y)dy\Big)\\
&= \frac{c_1}{\phi(r^{-2})}\Big(\int_{A(0,3r/4,r)} j(|z|)\Big(\int_{B_r^c} P_{B_r}(z,y)u(y)dy\Big)dz + \int_{B_r^c} j(|y|)u(y)dy\Big)\\
&= \frac{c_1}{\phi(r^{-2})}\Big(\int_{A(0,3r/4,r)} j(|z|)u(z)dz + \int_{B_r^c} j(|y|)u(y)dy\Big)\\
&\leq \frac{c_2}{\phi\big((2r/3)^{-2}\big)}\int_{B^c_{3r/4}} j(|y|)u(y)dy\,.
\end{aligned}$$

Similarly using Lemma 3.2.5, we also get $u(x) \geq \dfrac{c_3}{\phi(r^{-2})}\displaystyle\int_{B^c_{r/2}} j(|y|)\, u(y)\, dy$. Now applying Lemma 2.2.8, we have proved (3.2.14). (3.2.15) follows immediately from (3.2.14). □

For the remainder of the section, we fix C_* in Lemma 3.2.6 .

Lemma 3.2.7. *Suppose that $r \in (0,1)$. For nonnegative functions u_1, u_2 in $\mathbb{R}^d$ which are harmonic in B_r with respect to X, we have for every $0 < p < q/4 < 1/8$,*

$$\left(\sup_{B_{pr}} \frac{g_1}{g_2} - \inf_{B_{pr}} \frac{g_1}{g_2}\right) \leq \frac{C_*^2 - 1}{C_*^2 + 1}\left(\sup_{B_{qr}} \frac{u_1}{u_2} - \inf_{B_{qr}} \frac{u_1}{u_2}\right),$$

where $g_i(x) := \mathbb{E}_x[u_i(X_{\tau_{B_{2pr}}}) : X_{\tau_{B_{2pr}}} \in A(0, 2pr, qr)]$ for $i = 1,\ 2$.

Proof. For $a > 0$, we define $m_a = \inf_{B_a}(u_1/u_2)$ and $M_a = \sup_{B_a}(u_1/u_2)$. Let

$$f(x) := \mathbb{E}_x[(u_1 - m_{qr}u_2)(X_{\tau_{B_{2pr}}}) : X_{\tau_{B_{2pr}}} \in A(0, 2pr, qr)] = g_1(x) - m_{qr}g_2(x)$$

and

$$h(x) := \mathbb{E}_x[(M_{qr}u_2 - u_1)(X_{\tau_{B_{2pr}}}) : X_{\tau_{B_{2pr}}} \in A(0, 2pr, qr)] = M_{qr}g_2(x) - g_1(x),$$

then f and h are regular harmonic in B_{2pr} and nonnegative in $\mathbb{R}^d$. Thus by applying (3.2.15) to f and h, we get

$$\sup_{B_{pr}} \frac{g_1}{g_2} - m_{qr} = \sup_{B_{pr}} \frac{f}{g_2} \leq C_*^2 \inf_{B_{pr}} \frac{f}{g_2} = C_*^2 \left(\inf_{B_{pr}} \frac{g_1}{g_2} - m_{qr} \right)$$

and

$$M_{qr} - \inf_{B_{pr}} \frac{g_1}{g_2} = \sup_{B_{pr}} \frac{h}{g_2} \leq C_*^2 \inf_{B_{pr}} \frac{h}{g_2} = C_*^2 \left(M_{qr} - \sup_{B_{pr}} \frac{g_1}{g_2} \right).$$

By adding these inequalities, we proved the lemma. □

Now we are ready to prove the main result of this section. We prove the main result for the quotient of two harmonic functions in the next theorem. We closely follow the proof of [11, Lemma 8].

Theorem 3.2.8. *For every $\eta > 0$, there exists $a = a(\eta, \alpha, d, \ell) \in (0,1)$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0,1]$,*

$$\sup_{B(x_0,ar)} \frac{u_1}{u_2} \leq (1+\eta) \inf_{B(x_0,ar)} \frac{u_1}{u_2}$$

for nonnegative functions u_1 and u_2 in $\mathbb{R}^d$ which are harmonic in $B(x_0,r)$ with respect to X.

Proof. We assume $x_0 = 0$. We fix $r \in (0,1]$ and nonnegative functions u_1, u_2 in $\mathbb{R}^d$ which are harmonic in B_r with respect to X. Fix $\eta > 0$ and let

$$\varphi(t) := 1 + \frac{\eta}{2(C_*^2+1)} + \frac{C_*^2}{C_*^2+1}(t-1) \quad \text{for } t \geq 1$$

and $\varphi^1 := \varphi$, $\varphi^{l+1} := \varphi(\varphi^l)$ for $l = 1, 2, \cdots$. Then

$$\begin{aligned} \varphi^l(C_*^2) &= 1 + \frac{\eta}{2(C_*^2+1)} \sum_{i=0}^{l-1} (\frac{C_*^2}{C_*^2+1})^i + (\frac{C_*^2}{C_*^2+1})^l (C_*^2 - 1) \\ &\leq 1 + \frac{\eta}{2} + (\frac{C_*^2}{C_*^2+1})^l (C_*^2 - 1). \end{aligned}$$

Choose $l = l(C_*, \eta)$ large such that

$$(\frac{C_*^2}{C_*^2+1})^l\,(C_*^2-1) < \frac{\eta}{2} \quad \text{so that} \quad \varphi^l(C_*^2) < 1+\eta\,. \tag{3.2.16}$$

Also we choose $\varepsilon = \varepsilon(\eta)$ small enough so that

$$1+\frac{\eta}{C_*^2+1} \ge \big(C_*^3\,\varepsilon + (1+\varepsilon)\big)^2(1+\varepsilon)^2, \tag{3.2.17}$$

$$(1+C_*^2\varepsilon)^2 \le 1+\frac{\eta}{2(C_*^2+1)} \quad \text{and} \quad 1+C_*^2\,\varepsilon \le \frac{C_*^2}{C_*^2-1}. \tag{3.2.18}$$

Let $k = k(\varepsilon) \ge 3$ be the smallest integer such that $k > 1 + 1/\varepsilon^2$. We recall that $\delta = \delta(\varepsilon) > 0$ is the constant from (3.2.1) and fix it. Let $p_i := (\delta/6)^i/2$ for $i = 0, \cdots, lk-1$. For simplicity, we put $m_a := \inf_{B_a} u_1/u_2$ and $M_a := \sup_{B_a} u_1/u_2$.

Case 1. Suppose that the following holds for both $i = 1$ and 2; for every $0 \le m < lk$,

$$\begin{aligned} \Lambda_{rp_{m+1},rp_m}(u_i) &> \varepsilon\Lambda_{rp_m}(u_i)\,, \\ \text{i.e.,} \quad \int_{A(0,rp_{m+1},rp_m)} j(|y|)u_i(y)\,dy &> \varepsilon\int_{B^c_{rp_m}} j(|y|)u_i(y)\,dy\,. \end{aligned}$$

By the definition of k,

$$\begin{aligned} \Lambda_{2rp_{(j+1)k},rp_{jk}}(u_i) \ge \Lambda_{rp_{(j+1)k-1},rp_{jk}}(u_i) \;&=\; \sum_{m=0}^{k-2}\Lambda_{rp_{jk+m+1},rp_{jk+m}}(u_i) \\ \ge \varepsilon\sum_{m=0}^{k-2}\Lambda_{rp_{jk+m}}(u_i) \;&\ge\; (k-1)\varepsilon\,\Lambda_{rp_{jk}}(u_i) \;\ge\; \varepsilon^{-1}\Lambda_{rp_{jk}}(u_i) \end{aligned} \tag{3.2.19}$$

for $0 \le j \le l-1$.

For $i = 1$, 2 and $j = 1, \cdots, l-1$, we put

$$\begin{aligned} f_i^j(x) &:= \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{(j+1)k}}}}) : X_{\tau_{B_{2rp_{(j+1)k}}}} \in B^c_{rp_{jk}}] \\ &= \int_{B^c_{rp_{jk}}} P_{B_{2rp_{(j+1)k}}}(x,y)\, u_i(y)\, dy \end{aligned}$$

and

$$\begin{aligned} g_i^j(x) &:= \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{(j+1)k}}}}) : X_{\tau_{B_{2rp_{(j+1)k}}}} \in A(0, 2rp_{(j+1)k}, rp_{jk})] \\ &= \int_{A(0,2rp_{(j+1)k},rp_{jk})} P_{B_{2rp_{(j+1)k}}}(x,y)\, u_i(y)\, dy\,, \end{aligned}$$

then they are regular harmonic in $B_{2rp_{(j+1)k}}$ and $u_i = f_i^j + g_i^j$.

By (3.2.14) applied to $B_{rp_{(j+1)k}}$ in the first, and the facts that $f_i^j(x) = 0$ on $A(0, 2rp_{(j+1)k}, rp_{jk})$ and $f_i^j(x) = u_i(x)$ on $B^c_{rp_{jk}}$ in the second inequality, we have for $x \in B_{rp_{(j+1)k}}$,

$$\begin{aligned} f_i^j(x) &\le C_*\, \mathbb{E}_x[\tau_{B_{\frac{4}{3}rp_{(j+1)k}}}]\, \Lambda_{\frac{3}{2}rp_{(j+1)k}}(f_i^j) \\ &\le C_*\, \mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\, \Lambda_{rp_{jk}}(u_i) \qquad \text{for } j = 1, \cdots, l-1. \end{aligned}$$

Hence by (3.2.19), the fact that $g_i^j(x) = u_i(x)$ on $A(0, 2p_{(j+1)k}r, p_{jk}r)$ and (3.2.15) applied to $B_{rp_{(j+1)k}}$,

$$\begin{aligned} f_i^j(x) &\le C_*\, \varepsilon\, \mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\, \Lambda_{2rp_{(j+1)k},rp_{jk}}(u_i) \\ &= C_*\, \varepsilon\, \mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\, \Lambda_{2rp_{(j+1)k}}(g_i^j) \\ &\le C_*\, \varepsilon\, \mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\, \Lambda_{rp_{(j+1)k}}(g_i^j) \;\le\; C_*^2\, \varepsilon\, g_i^j(x) \end{aligned}$$

for $x \in B_{rp_{(j+1)k}}$ and $j = 1, \cdots, l-1$. Since $u_i(x) = f_i^j(x) + g_i^j(x)$ and $\dfrac{g_1^j}{f_2^j + g_2^j} \le \dfrac{u_1}{u_2} \le \dfrac{f_1^j + g_1^j}{g_2^j}$, we have

$$(1 + C_*^2\, \varepsilon)^{-1} \inf_{B_{rp_{(j+1)k}}} \frac{g_1^j}{g_2^j} \le m_{rp_{(j+1)k}} \le M_{rp_{(j+1)k}} \le (1 + C_*^2\, \varepsilon) \sup_{B_{rp_{(j+1)k}}} \frac{g_1^j}{g_2^j}$$

for $j = 1, \cdots, l-1$. Thus by Lemma 3.2.7,

$$
\begin{aligned}
&(C_*^2+1)\left((1+C_*^2\varepsilon)^{-1}M_{rp_{(j+1)k}} - (1+C_*^2\varepsilon)m_{rp_{(j+1)k}}\right) \\
&\le (C_*^2+1)\left(\sup_{B_{rp_{(j+1)k}}} \frac{g_1^j}{g_2^j} - \inf_{B_{rp_{(j+1)k}}} \frac{g_1^j}{g_2^j}\right) \le (C_*^2-1)(M_{rp_{jk}} - m_{rp_{jk}})
\end{aligned}
$$

for $j = 1, \cdots, l-1$. Multiplying by $(1+C_*^2\,\varepsilon)/(m_{rp_{(j+1)k}}(C_*^2+1))$ and using the obvious fact $m_{rp_{(j+1)k}} \ge m_{rp_{jk}}$, we obtain

$$
\frac{M_{rp_{(j+1)k}}}{m_{rp_{(j+1)k}}} \le (1+C_*^2\varepsilon)^2 + (1+C_*^2\varepsilon)\frac{C_*^2-1}{C_*^2+1}\left(\frac{M_{rp_{jk}}}{m_{rp_{jk}}} - 1\right).
$$

By the definition of φ and (3.2.18), $\dfrac{M_{rp_{(j+1)k}}}{m_{rp_{(j+1)k}}} \le \varphi\left(\dfrac{M_{rp_{jk}}}{m_{rp_{jk}}}\right)$. We already know that $\dfrac{M_{r/2}}{m_{r/2}} \le C_*^2$ by (3.2.15). And also by the monotonicity of φ and (3.2.16), we get

$$
\frac{M_{rp_{lk}}}{m_{rp_{lk}}} \;\le\; \varphi\left(\frac{M_{rp_{(l-1)k}}}{m_{rp_{(l-1)k}}}\right) \;\le\; \cdots \;\le\; \varphi^l\left(\frac{M_{r/2}}{m_{r/2}}\right) \;\le\; \varphi^l(C_*^2) \;<\; 1+\eta\,.
$$

Case 2. Suppose that there exists $m < lk$ such that for either $i = 1$ or 2,

$$
\begin{aligned}
\Lambda_{rp_{m+1},rp_m}(u_i) \quad &\le \quad \varepsilon\,\Lambda_{rp_m}(u_i)\,, \\
\text{i.e.,} \quad \int_{A(0,rp_{m+1},rp_m)} j(|y|)u_i(y)\,dy \quad &\le \quad \varepsilon\int_{B^c_{rp_m}} j(|y|)u_i(y)\,dy\,.
\end{aligned}
$$

Note that by (3.2.15),

$$
C_*^{-1}\frac{u_{3-i}(y)}{\Lambda_{rp_m}(u_{3-i})} \;\le\; \mathbb{E}_y[\tau_{B_{2rp_m}}] \;\le\; C_*\frac{u_i(y)}{\Lambda_{rp_m}(u_i)} \quad \text{for } y \in A(0, rp_{m+1}, rp_m).
$$

Hence by integrating on $A(0, rp_{m+1}, rp_m)$, we get

$$\frac{\Lambda_{rp_{m+1},rp_m}(u_{3-i})}{\Lambda_{rp_m}(u_{3-i})} \le C_*^2 \frac{\Lambda_{rp_{m+1},rp_m}(u_i)}{\Lambda_{rp_m}(u_i)} \le C_*^2 \varepsilon\,.$$

Thus

$$\Lambda_{rp_{m+1},rp_m}(u_i) \le C_*^2 \varepsilon\, \Lambda_{rp_m}(u_i) \quad \text{for both } \ i = 1 \text{ and } 2\,. \tag{3.2.20}$$

Let

$$\begin{aligned} f_i^m(x) = f_i(x) \;&:=\; \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{m+1}}}}) : X_{\tau_{B_{2rp_{m+1}}}} \in B^c_{rp_m}] \\ &= \int_{B^c_{rp_m}} P_{B_{2rp_{m+1}}}(x,y)\, u_i(y)\, dy \end{aligned}$$

and

$$\begin{aligned} g_i^m(x) = g_i(x) \;&:=\; \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{m+1}}}}) : X_{\tau_{B_{2rp_{m+1}}}} \in A(0, 2rp_{m+1}, rp_m)] \\ &= \int_{A(0,2rp_{m+1},rp_m)} P_{B_{2rp_{m+1}}}(x,y)\, u_i(y)\, dy\,, \end{aligned}$$

so that $u_i = f_i + g_i$. Since g_i is regular harmonic in $B_{2rp_{m+1}}$, by (3.2.14) we obtain for $x \in B_{rp_{m+1}}$,

$$g_i(x) \le C_*\, \mathbb{E}_x[\tau_{B_{\frac{4}{3}rp_{m+1}}}]\Lambda_{\frac{3}{2}rp_{m+1}}(g_i) \le C_*\, \mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_{m+1}}(g_i)\,.$$

Also since $g_i = 0$ on $\overline{B_{rp_m}}^c$ and $g_i = u_i$ on $A(0, 2rp_{m+1}, rp_m)$, we get

$$\begin{aligned} g_i(x) \;&\le\; C_*\, \mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_{m+1},rp_m}(g_i) \le C_*\, \mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_{m+1},rp_m}(u_i) \\ &\le\; \varepsilon\, C_*^3\, \mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_m}(u_i) \quad \text{for} \quad x \in B_{rp_{m+1}}\,. \end{aligned}$$

The last inequality comes from (3.2.20).

Then by (3.2.20), applying Lemma 3.2.1 to $f_i(x)$ and the fact that $\dfrac{f_1}{f_2 + g_2} \le$

$\dfrac{u_1}{u_2} \le \dfrac{f_1 + g_1}{f_2}$, we have for $x \in B_{rp_{m+1}}$,

$$\frac{(1+\varepsilon)^{-1}\Lambda_{rp_m}(u_1)}{\big((1+\varepsilon)+\varepsilon\, C_*^3\big)\Lambda_{rp_m}(u_2)} \le \frac{u_1(x)}{u_2(x)} \le \frac{\big((1+\varepsilon)+\varepsilon\, C_*^3\big)\Lambda_{rp_m}(u_1)}{(1+\varepsilon)^{-1}\Lambda_{rp_m}(u_2)}\,.$$

So by (3.2.17),

$$\frac{M_{rp_{lk}}}{m_{rp_{lk}}} \le \frac{M_{rp_{m+1}}}{m_{rp_{m+1}}} \le \big(\varepsilon\, C_*^3 + (1+\varepsilon)\big)^2 (1+\varepsilon)^2 \le 1 + \frac{\eta}{C_*^2+1} < 1+\eta\,.$$

In these two cases, we prove the theorem with $a = p_{lk}$. □

Theorem 3.2.9. *For every $\eta > 0$, there exists $a = a(\eta, \alpha, d, \ell) \in (0,1)$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0,1]$,*

$$\sup_{x\in B(x_0,ar)} u(x) \le (1+\eta) \inf_{x\in B(x_0,ar)} u(x)$$

for every nonnegative function u in $\mathbb{R}^d$ which is harmonic in $B(x_0,r)$ with respect to X.

Proof. Take $u_1 = u$ and $u_2 \equiv 1$ in Theorem 3.2.8. □

As a corollary of Theorem 3.2.9, we get the following.

Corollary 3.2.10. *There exists an increasing continuous function $\theta : (0,1) \to (0,\infty)$ with $\lim_{t\to 0}\theta(t) = 0$ such that for every $x_0 \in \mathbb{R}^d$, $R \in (0,1]$ and $r < R/2$,*

$$\sup_{x,y\in B(x_0,R/2),\,|x-y|<r} |u(x)-u(y)| \le \theta(|x-y|/r) \sup_{w\in B(x_0,R)} |u(w)|$$

for nonnegative function u in $\mathbb{R}^d$ which is harmonic in $B(x_0,R)$ with respect to X.

Proof. Without loss of generality, we assume $x_0 = 0$. For fixed $R \in (0,1]$ and r with $r < R/2$, let $x, y \in B_{R/2}$ be such that $|x-y| < r$ and $x, y \in$

$B(z, |x-y|) \subset B_R$ for some $z \in B_{R/2}$. For a nonnegative integer k, by Theorem 3.2.9 we can choose $a_{k+1} < a_k$ recurrently such that

$$\sup_{B(z, ra_k)} u \le (1 + 2^{-k-1}) \inf_{B(z, ra_k)} u \qquad \text{for } z \in B_{R/2}. \tag{3.2.21}$$

Define $a(\eta)$ using the linear interpolation as

$$a(\eta) = \begin{cases} a_k & \text{if} \quad \eta = 2^{-k} \\ \dfrac{a_k - a_{k+1}}{2^{-k} - 2^{-k-1}}\,\eta + 2a_{k+1} - a_k & \text{if} \quad 2^{-k-1} < \eta < 2^{-k}. \end{cases}$$

Then $a(\eta)$ is continuous and strictly increasing, so there exists an inverse function $\theta := a^{-1} : (0,1) \to (0, \infty)$, which is increasing and continuous.

Now we choose a nonnegative integer k such that $a_{k+1} \le \dfrac{|x-y|}{r} < a_k$, so that $2^{-k-1} \le \theta\big(\dfrac{|x-y|}{r}\big)$. Using this and (3.2.21), we get

$$\begin{aligned} \sup_{B(z,|x-y|)} u &\le \sup_{B(z, ra_k)} u \le (1 + 2^{-k-1}) \inf_{B(z, ra_k)} u \\ &\le \big(1 + \theta(\frac{|x-y|}{r})\big) \inf_{B(z, ra_k)} u \\ &\le \big(1 + \theta(\frac{|x-y|}{r})\big) \inf_{B(z,|x-y|)} u. \end{aligned}$$

Therefore

$$\begin{aligned} |u(x) - u(y)| &\le \sup_{B(z,|x-y|)} u - \inf_{B(z,|x-y|)} u \\ &\le \theta\big(\frac{|x-y|}{r}\big) \inf_{B(z,|x-y|)} u \le \theta\big(\frac{|x-y|}{r}\big) \sup_{B_R} u\,. \end{aligned}$$

□

Even though this corollary gives merely the continuity estimates, notice that the supremum is taken over the ball $B(x_0, R)$ and not the whole space $\mathbb{R}^d$ as in the existing literature (see [2, 3, 4, 13, 20, 21, 23, 35, 37]).

Chapter 4

Relative Fatou theorem

4.1 Hypothesis (A2) and its consequences

In this section, we assume $d \geq 2$. If $d = 2$, we will always assume the following.
(A2) : *There exists* $\gamma \in (0,1)$ *such that*

$$\liminf_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^\gamma} > 0.$$

Then by the criterion of Chung-Fuchs type, the process X is transient under this assumption (see [29, (13.3.1)]).

In this section, using Theorem 3.2.9 we prove the relative Fatou theorem. We first recall some results from [29, 30]. We will use

$$G(x,y) \;:=\; G(x-y) \;=\; \int_0^\infty p(t,x,y)\,dt$$

to denote the Green function of X. Since

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)}\, u(t)\,dt$$

where u is a density of the potential measure of S, G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$. G enjoys the following asymptotic property near the origin.

Theorem 4.1.1. ([29, Theorem 13.3.2])

$$G(x) \sim \frac{\alpha\Gamma\big((d-\alpha)/2\big)}{2^{\alpha+1}\pi^{d/2}\Gamma(1+\alpha/2)}\frac{1}{|x|^d\phi(|x|^{-2})} \qquad \text{as } |x| \to 0\,.$$

We recall that G_D is the Green function of X^D. G_D has the following interior lower bound estimates.

Lemma 4.1.2. ([30, Lemma 3.3]) *For every $L > 0$ and any bounded open set D, there exists $c = c(\alpha, d, \ell, diam(D), L) > 0$ such that for every $|x-y| \le L\big(\delta_D(x) \wedge \delta_D(y)\big)$,*

$$G_D(x,y) \,\ge\, c\,\frac{1}{|x-y|^d\phi(|x-y|^{-2})}\,.$$

Now we recall the following version of Harnack inequality for X. Note that, unlike Brownian motion, the next theorem does not require Harnack chain argument.

Theorem 4.1.3. ([30, Theorem 2.14]) *For $L > 0$, there exists a constant $c = c(\alpha, d, \ell, L) > 0$ such that the following is true: If $x_1, x_2 \in \mathbb{R}^d$ and $r \in (0,1)$ satisfy $|x_1 - x_2| < L\,r$, then for every nonnegative function u which is harmonic with respect to X in $B(x_1, r) \cup B(x_2, r)$, we have*

$$c^{-1}\,u(x_2) \,\le\, u(x_1) \,\le\, c\,u(x_2)\,.$$

From now on, we assume that D is a bounded κ-fat open set. We recall the definition of κ-fat open set.

Definition 4.1.4. *For $\kappa \in (0, 1/2]$, we say that an open set D in $\mathbb{R}^d$ is κ-fat if there exists $R > 0$ such that for each $Q \in \partial D$ and $r \in (0, R)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (R, κ) is called the characteristics of the κ-fat open set D.*

Note that all Lipschitz domains and all non-tangentially accessible domains (see [22] for the definition) are κ-fat. The boundary of a κ-fat open set may be not rectifiable, and in general, no regularity of its boundary can be inferred. A bounded κ-fat open set may be disconnected.

The following boundary Harnack principle is the main result in [28, 29].

Theorem 4.1.5. ([28, Theorem 4.8], [29, Theorem 13.4.22]) *Suppose that D is a κ-fat open set with the characteristics (R, κ). There exists a constant $c = c(\alpha, d, \ell, R, \kappa) > 1$ such that if $r \le R \wedge \frac{1}{4}$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have*

$$c^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le c \frac{u(A_r(Q))}{v(A_r(Q))} \qquad \text{for } x \in D \cap B(Q, \frac{r}{2}) .$$

4.2 Martin kernel

Let $x_0 \in D$ be fixed and set

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)} \qquad \text{for } x, y \in D \text{ and } y \neq x_0.$$

For each fixed $z \in \partial D$ and $x \in D$, let $M_D(x, z) := \lim_{D \ni y \to z} M_D(x, y)$, which exists by [28, Theorem 5.5]. For each $z \in \partial D$, set $M_D(x, z)$ to be zero for $x \in D^c$. M_D is called the Martin kernel of D with respect to X.

As a consequence of [28, Theorem 5.11], for every nonnegative harmonic function h with respect to X^D, there exists a unique finite measure ν on ∂D such that

$$h(x) = \int_{\partial D} M_D(x, z) \nu(dz) \qquad \text{for } x \in D.$$

ν is called the Martin measure of h.

The proof of the next result is similar to [16, Theorem 2.4] and [25, Lemma 3.2].

Lemma 4.2.1. *For each $z \in \partial D$, $M_D(\,\cdot\,, z)$ is bounded regular harmonic in $D \setminus B(z, \varepsilon)$ for every $\varepsilon > 0$.*

Proof. Fix $z \in \partial D$ and $\varepsilon > 0$, and let $h(x) := M_D(x, z)$ for $x \in \mathbb{R}^d$. Note that $G(x, y) \ge G_D(x, y)$. By Theorem 4.1.1, Lemma 4.1.2 and Theorem

4.1.5, there exist $c_1, c_2 > 0$ which depend on $\alpha, d, \ell, \kappa, R$ and $\mathrm{diam}(D)$ such that for every $x \in D \setminus B(z, \varepsilon/2)$,

$$
\begin{aligned}
h(x) = M_D(x,z) \;&=\; \lim_{D \ni y \to z} \frac{G_D(x,y)}{G_D(x_0,y)} \;\le\; c_1 \, \frac{G_D(x,A)}{G_D(x_0,A)} \\
&\le\; c_1 \, \frac{G(x,A)}{G_D(x_0,A)} \;\le\; c_2 \sup_{y \in D \setminus B(z,\varepsilon/2)} \frac{1}{|y-A|^d \, \phi(|y-A|^{-2}) G_D(x_0,A)} < \infty
\end{aligned}
$$

where $A := A_{\varepsilon/16}(z)$ (see Definition 4.1.4). Take an increasing sequence of smooth open sets $\{D_m\}_{m \ge 1}$ such that $\overline{D_m} \subset D_{m+1}$ and $\cup_{m=1}^{\infty} D_m = D \setminus B(z,\varepsilon)$. Set $\tau_m := \tau_{D_m}$ and $\tau_\infty := \tau_{D \setminus B(z,\varepsilon)}$. Then $\tau_m \uparrow \tau_\infty$ and $\lim_{m \to \infty} X_{\tau_m} = X_{\tau_\infty}$ by quasi-left continuity of X. Set

$$
E = \{\, \tau_m = \tau_\infty \;\text{ for some } m \ge 1\}
$$

and N be the set of irregular boundary points of D. Since X is symmetric, by [8, (VI.4.6), (VI.4.10)] we get

$$
\mathbb{P}_x(X_{\tau_\infty} \in N) = 0 \quad \text{for } \; x \in D. \tag{4.2.1}
$$

We also know from [28, Lemma 5.9(i)] that if $w \in \partial D, w \neq z$ and w is a regular boundary point, then $h(x) \to 0$ as $x \to w$ so that h is continuous on $\overline{D \setminus B(z,\varepsilon)} \setminus N$. Since h is bounded on $\mathbb{R}^d \setminus B(z,\varepsilon/2)$, by the bounded convergence theorem and (4.2.1), we have

$$
\begin{aligned}
\lim_{m\to\infty} \mathbb{E}_x\left[h(X_{\tau_m}) \, ; \, \tau_m < \tau_\infty \right] \;&=\; \lim_{m\to\infty} \mathbb{E}_x\Big[h(X_{\tau_m}) 1_{\overline{D \setminus B(z,\varepsilon)} \setminus N}(X_{\tau_m}) \, ; \, \tau_m < \tau_\infty \Big] \\
&=\; \mathbb{E}_x\Big[h(X_{\tau_\infty}) 1_{\overline{D \setminus B(z,\varepsilon)} \setminus N}(X_{\tau_\infty}) \, ; \, E^{\,c} \Big] \\
&=\; \mathbb{E}_x\left[h(X_{\tau_\infty}) \, ; \, E^{\,c} \right].
\end{aligned} \tag{4.2.2}
$$

Since $\tau_m \uparrow \tau_\infty$ and $\{\tau_m = \tau_\infty\} = \{\tau_n = \tau_\infty, \, n \ge m\} \uparrow E$ as $m \to \infty$, by

(4.2.2) and the monotone convergence theorem,

$$\begin{aligned} h(x) &= \lim_{m\to\infty} \mathbb{E}_x[h(X_{\tau_m})] \\ &= \lim_{m\to\infty} \mathbb{E}_x[h(X_{\tau_m})\,;\,\tau_m < \tau_\infty] + \lim_{m\to\infty} \mathbb{E}_x[h(X_{\tau_\infty})\,;\,\tau_m = \tau_\infty] \\ &= \mathbb{E}_x[h(X_{\tau_\infty})\,;\,E^{\,c}\,] + \mathbb{E}_x[h(X_{\tau_\infty})\,;\,E\,] \,=\, \mathbb{E}_x[h(X_{\tau_\infty})]\,. \end{aligned}$$

□

Throughout this thesis, $\mathcal{F}_t$ is the augmented right continuous σ-field generated by X^D_t. For a positive harmonic function h with respect to X^D, we let $(\mathbb{P}^h_x, X^h_t)$ be the h-transform of $(\mathbb{P}_x, X^D_t)$, that is,

$$\mathbb{P}^h_x(A) := \mathbb{E}_x\left[\frac{h(X^D_t)}{h(x)}; A\right] \qquad \text{if } A \in \mathcal{F}_t.$$

When $h(\cdot) = M_D(\cdot, z)$, we use the notation $(\mathbb{P}^z_x, X^z_t) := (\mathbb{P}^h_x, X^h_t)$ so that $(\mathbb{P}^z_x, X^z_t)$ is $M_D(\cdot, z)$-transform of $(\mathbb{P}_x, X^D_t)$.

Let τ^z_D be the life time of X^z. The proof of the next result is similar to [25, Theorem 3.3] and [16, Theorem 3.17].

Theorem 4.2.2.

$$\mathbb{P}^z_x\Big(\lim_{t\uparrow\tau^z_D} X^z_t = z,\ \tau^z_D < \infty\Big) = 1 \quad \textit{for every } x \in D \textit{ and } z \in \partial D.$$

Proof. Recall that the following 3G inequality proved in [26, Theorem 3.10]; For $x, y, w \in D$,

$$\frac{G_D(x,y)G_D(y,w)}{G_D(x,w)} \le c_1\left(\frac{1}{\phi(|x-y|^{-2})\,|x-y|^d} + \frac{1}{\phi(|y-w|^{-2})\,|y-w|^d}\right).$$

Thus for $x, y, w \in D$,

$$\frac{G_D(x,y)M_D(y,w)}{M_D(x,w)} \le c_1\left(\frac{1}{\phi(|x-y|^{-2})\,|x-y|^d} + \frac{1}{\phi(|y-w|^{-2})\,|y-w|^d}\right).$$

Using this, we have

$$\begin{aligned}
\mathbb{E}_x^z[\tau_D^z] \;&=\; \mathbb{E}_x^z \int_0^\infty 1_{\{t<\tau_D^z\}}\, dt \;=\; \frac{1}{M_D(x,z)} \int_0^\infty \mathbb{E}_x\left[M_D(X_t^D, z);\, t < \tau_D\right] dt \\
&= \int_D \frac{G_D(x,y) M_D(y,z)}{M_D(x,z)}\, dy \;\le\; 2\, c_1 \sup_{x\in\overline{D}} \int_D \frac{1}{\phi(|x-y|^{-2})|x-y|^d}\, dy,
\end{aligned}$$

which is finite by **(A1)**. Thus there exists a positive constant $c_2 = c_2(\alpha, d, \ell, D) < \infty$ such that

$$\sup_{x\in D, z\in\partial D} \mathbb{E}_x^z[\tau_D^z] \;\le\; c_2.$$

Therefore $\mathbb{P}_x^z(\tau_D^z < \infty) = 1$ for every $x \in D$ and $z \in \partial D$.

Let $r_m = 1/2^m$, $B_m = B(z, r_m)$ and $D_m = D \setminus \overline{B_m}$. We set $T_m := T_{B_m}$ and $R_m := \tau_{B_m \cap D}$. By Lemma 4.2.1 and [16, Lemma 3.12], we get

$$\begin{aligned}
M_D(x,z) \;&=\; \mathbb{E}_x[M_D(X_{\tau_{D_m}}, z)] \\
&=\; \mathbb{E}_x[M_D(X_{\tau_{D_m}}, z); T_m < \tau_D] + \mathbb{E}_x[M_D(X_{\tau_{D_m}}, z); T_m \ge \tau_D] \\
&=\; \mathbb{E}_x[M_D(X_{T_m}, z); T_m < \tau_D] \\
&=\; \int_D M_D(y,z) 1_{\{T_m<\tau_D\}} \mathbb{P}_x(X_{T_m} \in dy) \\
&=\; M_D(x,z) \mathbb{P}_x^z(T_m < \tau_D).
\end{aligned}$$

Therefore

$$\mathbb{P}_x^z(T_m < \tau_D) = 1. \tag{4.2.3}$$

Now we claim that $\mathbb{P}_x^z\big[\lim_{t\uparrow\tau_D^z} X_t^z = z\big] = 1$. Let $A_k := \sup_{B_k^c \cap D} M_D(\cdot, z)$, which is finite by Lemma 4.2.1. By the strong Markov property, the definition of $M_D(\cdot, z)$-transform and [16, Lemmas 3.12 and 3.13], we get

$$\begin{aligned}
\mathbb{P}_x^z\left(T_m < \tau_D,\, R_k \circ \theta_{T_m} < \tau_D\right) &= \mathbb{E}_x^z[\mathbb{P}^z_{X_{T_m}}(R_k < \tau_D); T_m < \tau_D] \\
&= \frac{1}{M_D(x,z)} \mathbb{E}_x[M_D(X_{T_m}, z)\mathbb{P}^z_{X_{T_m}}(R_k < \tau_D); T_m < \tau_D] \\
&= \frac{1}{M_D(x,z)} \mathbb{E}_x\big[\mathbb{E}_{X_{T_m}}[M_D(X_{R_k}, z); R_k < \tau_D]; T_m < \tau_D\big] \\
&\le \frac{A_k}{M_D(x,z)}\, \mathbb{P}_x(T_m < \tau_D) \quad \text{for } k < m.
\end{aligned}$$

And by the quasi-left continuity,

$$\lim_{m\to\infty} \mathbb{P}_x(T_m < \tau_D) \le \mathbb{P}_x(\lim_{m\to\infty} T_m \le \tau_D) \le \mathbb{P}_x(T_{\{z\}} \le \tau_D) = 0.$$

The last equality comes from the fact that one point set is essentially polar if $d \ge 2$ (see [6, chapter II.3]). Then for each $k \ge 1$, there exists a subsequence $\{m_j\}$ such that

$$\sum_{j=1}^{\infty} \mathbb{P}_x^z\left(T_{m_j} < \tau_D,\, R_k^z \circ \theta_{T^z_{m_j}} < \tau_D^z\right) \,<\, \infty\,,$$

so that by Borel-Cantelli lemma (see [1, Proposition 1.1]) we have

$$\mathbb{P}_x^z\left(\left[T_{m_j} < \tau_D,\, R_k^z \circ \theta_{T^z_{m_j}} < \tau_D^z\right] \text{ infinitely often }\right) = 0,$$

i.e.,

$$\mathbb{P}_x^z\left(\left[T_{m_j} \ge \tau_D \text{ or } R_k^z \circ \theta_{T^z_{m_j}} \ge \tau_D^z\right] \text{ except finitely many }\right) = 1. \tag{4.2.4}$$

(4.2.4) and (4.2.3) imply that for each $k \ge 1$ and $\mathbb{P}_x^z$-a.e. ω, there exists $N = N(\omega) < \infty$ such that $R_k \circ T_{m_j} \ge \tau_D$ for $j \ge N$, i.e.,

$$X_t(\omega) \in B(z, r_k) \text{ for all } t \in [T_N, \tau_D).$$

For each k, let $N(k)$ be the smallest N which satisfies the above. Then $T_{N(k)} \uparrow \tau_D$ as $k \to \infty$. This proves that $X_t \to z$ as $t \to \tau_D$ for $\mathbb{P}_x^z$-a.e. ω. $\square$

The following result is a simple consequence of Theorem 4.2.2.

Proposition 4.2.3. *Let h be a positive harmonic function with respect to X^D with Martin measure ν. Then*

$$\mathbb{P}_x^h\Big(A \cap \big\{ \lim_{t\uparrow\tau_D^h} X_t^h \in K\big\}\Big) = \frac{1}{h(x)} \int_K M_D(x,z)\mathbb{P}_x^z(A)\nu(dz)$$

for every $x \in D$, $A \in \mathcal{F}_{\tau_D}$ and Borel subset K of ∂D.

Proof. This proof is similar to [25, Proposition 3.5]. Since Theorem 4.2.2 implies that $\mathbb{P}_x^a(\lim_{t\uparrow\tau_D} X_t \in K) = 1_K(a)$ for every $x \in D$ and Borel subset $K \subset \partial D$, we have that for $x \in D$,

$$\mathbb{P}_x^h\Big(\lim_{t\uparrow\tau_D^h} X_t^h \in K\Big) = \frac{1}{h(x)} \int_K M_D(x,w)\nu(dw). \tag{4.2.5}$$

Take a sequence of open sets $\{D_m\}$ such that $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^\infty D_m = D$. Put $\tau_m = \tau_{D_m}$ and fix $A \in \mathcal{F}_{\tau_m}$. Then by the definition of $M_D(x,z)$-transform, Fubini theorem, (4.2.5) and the strong Markov property for the conditional expectation, we get for every Borel subset $K \subset \partial D$,

$$\begin{aligned}
& \frac{1}{h(x)} \int_K M_D(x,z)\, \mathbb{P}_x^z(A)\, \nu(dz) \\
= \ & \frac{1}{h(x)} \int_K M_D(x,z)\, \mathbb{E}_x\left[\frac{M_D(X_{\tau_m}^D, z)}{M_D(x,z)}; A\right] \nu(dz) \\
= \ & \frac{1}{h(x)}\, \mathbb{E}_x \left[\int_K M_D(X_{\tau_m}^D, z)\, \nu(dz); A\right] \\
= \ & \frac{1}{h(x)}\, \mathbb{E}_x \left[h(X_{\tau_m}^D)\, \mathbb{P}_{X_{\tau_m}}^h \big(\lim_{t\uparrow\tau_D^h} X_t^h \in K\big); A\right] \\
= \ & \mathbb{E}_x^h \left[\mathbb{P}_{X_{\tau_m}}^h \big(\lim_{t\uparrow\tau_D^h} X_t^h \in K\big); A\right] \\
= \ & \mathbb{P}_x^h \left(A \cap \{\lim_{t\uparrow\tau_D^h} X_t^h \in K\}\right).
\end{aligned}$$

Let $m \to \infty$, then

$$\frac{1}{h(x)} \int_K M_D(x,z)\, \mathbb{P}_x^z(A)\, \nu(dz) \;=\; \mathbb{P}_x^h \left(A \cap \{ \lim_{t \uparrow \tau_D^h} X_t^h \in K \} \right) \qquad (4.2.6)$$

for $A \in \bigcap_{m \ge 1} \mathcal{F}_{\tau_m}$. By the monotone class theorem (see [19, p. 277]), (4.2.6) holds for $A \in \mathcal{F}_{\tau_D}$. $\square$

4.3 Proof of the relative Fatou theorem

Definition 4.3.1. *$A \in \mathcal{F}_{\tau_D}$ is shift-invariant if whenever $T < \tau_D$ is a stopping time, $1_A \circ \theta_T = 1_A$ $\mathbb{P}_x$-a.s. for every $x \in D$.*

Using [28, Theorem 5.11], the proof of the next proposition is the same as the one in [25, Proposition 3.7] (see also [1, Theorem III.2.9]).

Proposition 4.3.2. (0-1 law) *If A is shift-invariant, then $x \mapsto \mathbb{P}_x^z(A)$ is a constant function which is either 0 or 1.*

Proof. Let $T < \tau_D$ be a stopping time, then by the strong Markov property we get

$$\mathbb{P}_x^z(A) \;=\; \mathbb{P}_x^z(A \circ \theta_T) \;=\; \mathbb{E}_x^z\big(\mathbb{P}_{X_T}^z(A)\big) \;=\; \mathbb{E}_x \left[\frac{M_D(X_T^D, z)}{M_D(x,z)}\, \mathbb{P}_{X_T}^z(A) \right].$$

Therefore $x \mapsto M_D(x,z)\, \mathbb{P}_x^z(A)$ is harmonic in D and it is bounded above by $M_D(x,z)$ because $\mathbb{P}_x^z(A) \le 1$. Since $x \mapsto M_D(x,z)$ is a minimal harmonic function (see [28, Theorem 5.11]), $\mathbb{P}_x^z(A)$ is a constant.

Take a sequence of open sets $\{D_m\}$ such that $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = D$. Put $\tau_m = \tau_{D_m}$ and fix $B \in \mathcal{F}_{\tau_m}$. Since $x \mapsto \mathbb{P}_x^z(A)$ is a constant, we get

$$\mathbb{P}_x^z(A \cap B) \;=\; \mathbb{P}_x^z\big((A \circ \theta_{\tau_m}); B\big) \;=\; \mathbb{E}_x^z[\, \mathbb{P}_{X_{\tau_m}}^z(A); B\,] \;=\; \mathbb{P}_x^z(A)\, \mathbb{P}_x^z(B).$$

Thus $\mathbb{P}_x^z(A \cap B) = \mathbb{P}_x^z(A)\, \mathbb{P}_x^z(B)$ for $B \in \mathcal{F}_{\tau_m}$. Let $m \to \infty$ and put $B = A$. Then we have $\mathbb{P}_x^z(A) = \big(\mathbb{P}_x^z(A)\big)^2$, hence $\mathbb{P}_x^z(A)$ is either 0 or 1. $\square$

From now on, we use notations $T_B := \inf\{t > 0 :\ X_t \in B\}$, $T_B^z := \inf\{t > 0 :\ X_t^z \in B\}$ and $B_y^\lambda := B(y, \lambda \delta_D(y))$ for the convenience.

Proposition 4.3.3. *There exists $c = c(\alpha, \ell, D) > 1$ such that if $0 < \lambda < 1/2$ and $x, y \in D$ with $|y - x| > 2\delta_D(y)$, then*

$$\mathbb{P}_x\left(T_{B_y^\lambda} < \tau_D\right) \geq c\ G_D(x,y)\lambda^d \delta_D(y)^d \phi\big((2\lambda\delta_D(y))^{-2}\big)\,.$$

Proof. Fix $\lambda \in (0, 1/2)$ and $x, y \in D$ with $|y - x| > 2\delta_D(y)$. Since $x \notin B(y, \delta_D(y))$, by Theorem 4.1.3 we get

$$\mathbb{E}_x\Big[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\Big] = \int_{B_y^\lambda} G_D(x,z)dz \;\geq\; c_1 G_D(x,y)\lambda^d\delta_D(y)^d\,. \qquad (4.3.1)$$

On the other hand, by the strong Markov property,

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\Big] &= \mathbb{E}_x\Big[\mathbb{E}_{X_{T_{B_y^\lambda}}}\Big[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\Big] : T_{B_y^\lambda} < \tau_D\Big] \\ &\leq \mathbb{P}_x\big(T_{B_y^\lambda} < \tau_D\big) \sup_{w \in \overline{B_y^\lambda}} \mathbb{E}_w\Big[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\Big]\,. \qquad (4.3.2)\end{aligned}$$

Note that since $0 < \lambda\,\delta_D(y) \leq \mathrm{diam}(D)$, by (3.2.6) and Theorem 4.1.1 we obtain for every $w \in \overline{B_y^\lambda}$,

$$\begin{aligned}\mathbb{E}_w\Big[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\Big] &\leq \int_{B_y^\lambda} G(w-v)dv \;\leq\; c_2\int_{B_y^\lambda} \frac{dv}{|w-v|^d\phi(|w-v|^{-2})} \\ &\leq c_2 \int_{\{|w-v| \leq 2\lambda\delta_D(y)\}} \frac{dv}{|w-v|^d\phi(|w-v|^{-2})} \\ &= c_3 \int_0^{2\lambda\delta_D(y)} \frac{1}{s\phi(s^{-2})}ds \;\leq\; c_4\,\frac{1}{\phi\big((2\lambda\delta_D(y))^{-2}\big)}\,.\end{aligned}$$

Combining this with (4.3.1)–(4.3.2), we finish the proof. □

Now we define the Stolz open set for κ-fat open set D with the characteristics (R, κ).

Definition 4.3.4. *For $z \in \partial D$ and $\beta > (1-\kappa)/\kappa$, let*

$$A_z^\beta := \{y \in D\,;\, \delta_D(y) < R \wedge \big(\delta_D(x_0)/3\big) \text{ and } |y - z| < \beta\,\delta_D(y)\}.$$

We call A_z^β the Stolz open set for D at z with the angle β.

Since $\beta > (1-\kappa)/\kappa$, there exists a sequence $\{y_k\}_{k\geq 1} \subset A_z^\beta$ such that $\lim_{k\to\infty} y_k = z$ (see [25, Lemma 3.9]).

Proposition 4.3.5. *Given $\beta > (1-\kappa)/\kappa$ and $x \in D$, there exists $c = c(\alpha, \beta, D, x) > 0$ such that for every $z \in \partial D$, $\lambda \in (0, 1/2)$ and $y \in A_z^\beta$ with $\delta_D(y) \leq \frac{1}{2}|x-y| \wedge \delta_D(x)$, we have*

$$\mathbb{P}_x^z\Big(T^z_{B_y^\lambda} < \tau^z_D\Big) \;>\; c\,\lambda^d\,\frac{\phi\big((2\lambda\delta_D(y))^{-2}\big)}{\phi\big((\delta_D(y)/8)^{-2}\big)}.$$

Proof. Fix $\beta > (1-\kappa)/\kappa$, $z \in \partial D$, $x \in D$, $\lambda \in (0, 1/2)$ and $y \in A_z^\beta$ with $\delta_D(y) \leq \frac{1}{2}|x-y| \wedge \delta_D(x)$. Let $z_1 := A_{\delta_D(y)/8}(z)$ so that $B(z_1, \kappa\,\delta_D(y)/8) \subset B(z, \delta_D(y)/8) \cap D$ and fix $z_2 \in \partial B(y, \delta_D(y)/8)$. Since $M_D(\cdot, z)$ is a harmonic function with respect to X in D (Lemma 4.2.1), by the Harnack principle (Theorem 4.1.3) and Proposition 4.3.3 we have

$$\begin{aligned}
\mathbb{P}_x^z\left(T^z_{B_y^\lambda} < \tau^z_D\right) &= \mathbb{E}_x\left[\frac{M_D(X_{T_{B_y^\lambda}}, z)}{M_D(x,z)}\,;\, T_{B_y^\lambda} < \tau_D\right] \\
&\geq c_1\,\mathbb{P}_x\left(T_{B_y^\lambda} < \tau_D\right)\,\frac{M_D(y,z)}{M_D(x,z)} \\
&\geq c_2\,G_D(x,y)\lambda^d\delta_D(y)^d\phi\big((2\lambda\delta_D(y))^{-2}\big)\lim_{D\ni w\to z}\frac{G_D(y,w)}{G_D(x,w)} \\
&\geq c_3\,G_D(x,y)\lambda^d\delta_D(y)^d\phi\big((2\lambda\delta_D(y))^{-2}\big)\frac{G_D(y,z_1)}{G_D(x,z_1)}.
\end{aligned}$$

The last inequality comes from Theorem 4.1.5 because $|y-z| \wedge |x-z| > \delta_D(y)/2$. We see that $\delta_D(z_1) \geq \kappa\delta_D(y)/8 > \delta_D(y)/\big(8(\beta+1)\big)$, $\delta_D(z_2) > \delta_D(y)/2$ and $|z_2 - y| = \delta_D(y)/8$. Moreover using our assumptions that $\delta_D(y) \leq \delta_D(x)$ and $|x-y| \geq 2\delta_D(y)$, we have

$$\begin{aligned}
|z_2 - x| &\geq |x-y| - |y-z_2| \geq 2\delta_D(y) - \frac{\delta_D(y)}{8} > \delta_D(y)\,, \\
|z_1 - x| &\geq |x-z| - |z-z_1| \geq \delta_D(x) - \frac{\delta_D(y)}{8} > \frac{\delta_D(y)}{2}
\end{aligned}$$

and
$$|z_1 - y| \ge |y-z| - |z_1 - z| \ge \delta_D(y) - \frac{\delta_D(y)}{8} > \frac{\delta_D(y)}{2}.$$
Thus $G_D(y,\cdot)$ and $G_D(x,\cdot)$ are harmonic functions in $B\big(z_1,\, 8^{-1}(\beta+1)^{-1}\delta_D(y)\big) \cup B\big(z_2,\, 8^{-1}(\beta+1)^{-1}\delta_D(y)\big)$. Since $|z_1 - z_2| \le |z_1 - z| + |z-y| + |y-z_2| < (4^{-1}+\beta)\,\delta_D(y)$, by Theorem 4.1.3 we have $G_D(y,z_1) \ge c_4 G_D(y,z_2)$ and $G_D(x,z_1) \le c_5 G_D(x,z_2) \le c_6 G_D(x,y)$. On the other hand, by Lemma 4.1.2 and (3.2.2), we get
$$G_D(y,z_2) \ge c_7 \frac{1}{|y-z_2|^d \phi(|y-z_2|^{-2})} \ge c_8 \frac{1}{\delta_D(y)^d \phi\big((\delta_D(y)/8)^{-2}\big)}.$$
Combining these observations, we prove the proposition. □

Now we are ready to show the relative Fatou theorem for the harmonic function with respect to X in D. The proof is similar to the proof of [25, Theorem 3.13]. But, since we state a slightly more general version, we spell out detail for the reader's convenience.

Theorem 4.3.6. *Let h be a positive harmonic function with respect to X^D with the Martin measure ν. If $x \in D$ and u is a nonnegative function which is harmonic in D with respect to X, then for ν-a.e. $z \in \partial D$, $\lim_{t\uparrow\tau_D^z} u(X_t^z)/h(X_t^z)$ exists and is finite $\mathbb{P}_x^z$-a.s. Moreover, for every $x \in D$ and every $\beta > (1-\kappa)/\kappa$,*
$$\lim_{t\uparrow\tau_D^z} \frac{u(X_t^z)}{h(X_t^z)} = \lim_{A_z^\beta \ni y \to z} \frac{u(y)}{h(y)} \quad \mathbb{P}_x^z\text{-a.s.} \tag{4.3.3}$$
In particular for ν-a.e. $z \in \partial D$,
$$\lim_{A_z^\beta \ni y\to z} \frac{u(y)}{h(y)} \text{ exists for every } \beta > \frac{1-\kappa}{\kappa}. \tag{4.3.4}$$

Proof. Without loss of generality, we assume $\nu(\partial D) = 1$ and fix $x \in D$. Note that u is a nonnegative and continuous superharmonic function with respect to X^D, i.e., for $x \in B$, $u(x) \ge \mathbb{E}_x\left[u(X^D_{\tau_B})\right]$ for every open set B whose closure is a compact subset of D. Since X^D is a Hunt process and u is nonnegative and continuous superharmonic with respect to X^D, u is excessive with respect to X^D (see [8, Corollary II.5.3] and the second part in the proof of [1, Proposition II.6.7]). In particular, $\mathbb{E}_w[u(X_t^D)] \le u(w)$ for every $w \in D$. So by the Markov property for the conditional process (for example, see [17, Chapter 11]), we have for every $t, s > 0$

$$\mathbb{E}_x^h\left[\frac{u(X_{t+s}^h)}{h(X_{t+s}^h)} \,\Big|\, \mathcal{F}_s\right] = \mathbb{E}_{X_s^h}^h\left[\frac{u(X_t^h)}{h(X_t^h)}\right] = \frac{1}{h(X_s^h)}\,\mathbb{E}_{X_s^h}\left[u(X_t^D)\right] \le \frac{u(X_s^h)}{h(X_s^h)}\,.$$

Therefore we see that $u(X_t^h)/h(X_t^h)$ is a nonnegative supermartingale with respect to $\mathbb{P}_x^h$, and so the martingale convergence theorem gives $\lim_{t\uparrow\tau_D^h}\dfrac{u(X_t^h)}{h(X_t^h)}$ exists and is finite $\mathbb{P}_x^h$-a.s. Thus by Proposition 4.2.3,

$$\mathbb{P}_x^z\left(\lim_{t\uparrow\tau_D^z}\frac{u(X_t^z)}{h(X_t^z)} \text{ exists and is finite}\right) = 1 \tag{4.3.5}$$

for ν-a.e. $z \in \partial D$.

Fix $z \in \partial D$ satisfying (4.3.5) and $\beta > (1-\kappa)/\kappa$. By (2.2.1) and Proposition 4.3.5, for every sequence $\{y_k\}_{k=1}^\infty \subset A_z^\beta$ converging to z,

$$\mathbb{P}_x^z\Big(T_{B_{y_k}^\lambda}^z < \tau_D^z \;\text{ i.o.}\Big) \ge \liminf_{k\to\infty}\mathbb{P}_x^z\Big(T_{B_{y_k}^\lambda}^z < \tau_D^z\Big) > 0$$

for every $\lambda \in (0, 1/2)$. Since $\{T_{B_{y_k}^\lambda}^z < \tau_D^z \;\text{ i.o.}\}$ is shift-invariant, by Proposition 4.3.2,

$$\mathbb{P}_x^z\left(X_t^z \text{ hits infinitely many } B_{y_k}^\lambda\right) \;=\; \mathbb{P}_x^z\Big(T_{B_{y_k}^\lambda}^z < \tau_D^z \;\text{ i.o.}\Big) \;=\; 1 \tag{4.3.6}$$

for every $\lambda \in (0, 1/2)$.

Now let

$$m := \liminf_{A_z^\beta \ni y\to z}\frac{u(y)}{h(y)} \quad \text{ and } \quad l := \limsup_{A_z^\beta \ni y\to z}\frac{u(y)}{h(y)}.$$

First we note that $l < \infty$. If not, for any $M > 1$ there exists a sequence $\{x_k\}_{k=1}^\infty \subset A_z^\beta$ such that $\dfrac{u(x_k)}{h(x_k)} > 4M$ and $x_k \to z$. By Theorem 3.2.9, there exists $\lambda_1 = \lambda_1(M, \alpha, d, \ell) > 0$ such that $\dfrac{u(w)}{h(w)} \ge M^2(M+1)^{-2}\,\dfrac{u(x_k)}{h(x_k)} > M$ for every $w \in B^{\lambda_1}_{x_k}$. Thus by (4.3.6), we have $\lim_{t\uparrow\tau^z_D} \dfrac{u(X^z_t)}{h(X^z_t)} > M$ $\mathbb{P}^z_x$-a.s. for every $M > 1$, which is a contradiction to (4.3.5). Also if $l = 0$, then $0 \le m \le l = 0$ so the theorem is clear. So we assume $0 < l < \infty$.

For given $\varepsilon > 0$, choose sequences $\{y_k\}_{k=1}^{\infty} \cup \{z_k\}_{k=1}^{\infty} \subset A^\beta_z$ such that $\dfrac{u(y_k)}{h(y_k)} > (1+\varepsilon)^{-1}\, l$, $\dfrac{u(z_k)}{h(z_k)} < m+\varepsilon$ and $y_k, z_k \to z$. By Theorem 3.2.9, there is $\lambda_2 = \lambda_2(\varepsilon, \alpha, d, \ell) > 0$ such that

$$\frac{u(w)}{h(w)} \;\ge\; \frac{u(y_k)}{(1+\varepsilon)^2\, h(y_k)} \;>\; \frac{l}{(1+\varepsilon)^3} \quad \text{for every } w \in B^{\lambda_2}_{y_k} \tag{4.3.7}$$

and

$$\frac{u(w)}{h(w)} \;\le\; (1+\varepsilon)^2\, \frac{u(z_k)}{h(z_k)} \;<\; (1+\varepsilon)^2 (m+\varepsilon) \quad \text{for every } w \in B^{\lambda_2}_{z_k}. \tag{4.3.8}$$

Applying (4.3.5)–(4.3.6) to (4.3.7)–(4.3.8) and letting $\varepsilon \downarrow 0$, we obtain both (4.3.3) and (4.3.4). □

Remark 4.3.7. Since constant functions in $\mathbb{R}^d$ are harmonic with respect to X in D, one can easily see that the above theorem is also true for every harmonic function u with respect to X in D either bounded below or above.

If u and h are harmonic functions in D and u/h is bounded, then u can be recovered from non-tangential limit values of u/h.

Theorem 4.3.8. *If u is a harmonic function in D with respect to X and u/h is bounded for a positive harmonic function h in D with respect to X^D with the Martin measure ν, then*

$$u(x) \;=\; h(x)\, \mathbb{E}^h_x\Big[\varphi_u\Big(\lim_{t\uparrow\tau^h_D} X^h_t\Big)\Big] \qquad \textit{for every } x \in D,$$

where $\varphi_u(z) := \lim_{A^\beta_z \ni x \to z} u(x)/h(x)$ for $\beta > (1-\kappa)/\kappa$ which is well-defined for ν-a.e. $z \in \partial D$. If we further assume that u is positive in D, then $\varphi_u(z)$ is the Radon-Nikodym derivative of the (unique) Martin measure of u with respect to ν.

Proof. This proof is similar to [25, Theorem 3.18]. We can assume that u is positive harmonic in D. Take a sequence of open sets $\{D_m\}$ such that $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = D$. Put $\tau_m = \tau_{D_m}$. Then Theorems 4.2.2 and 4.3.6 imply that

$$\begin{aligned} 1 &= \mathbb{P}_x^z \left(\lim_{m\to\infty} \frac{u(X_{\tau_m})}{h(X_{\tau_m})} = \lim_{t\uparrow\tau_D^z} \frac{u(X_t^z)}{h(X_t^z)} = \lim_{A_z^\beta \ni y \to z} \frac{u(y)}{h(y)} \right) \\ &= \mathbb{P}_x^z \left(\lim_{m\to\infty} \frac{u(X_{\tau_m})}{h(X_{\tau_m})} = \varphi_u(z),\ \lim_{t\uparrow\tau_D^z} X_t = z \right) \end{aligned}$$

for ν-a.e. $z \in \partial D$. Then by Proposition 4.2.3, we get

$$1 = \mathbb{P}_x^h \left(\lim_{m\to\infty} \frac{u(X_{\tau_m})}{h(X_{\tau_m})} = \varphi_u(z),\ \lim_{t\uparrow\tau_D^h} X_t^h = z \right),$$

i.e.,

$$\lim_{m\to\infty} \frac{u(X_{\tau_m}^h)}{h(X_{\tau_m}^h)} = \varphi_u\big(\lim_{t\uparrow\tau_D^h} X_t^h\big) \quad \mathbb{P}_x^h\text{-a.s.} \tag{4.3.9}$$

Note that by the definition of h-transform, u/h is harmonic with respect to $\mathbb{P}_x^h$. Then by the bounded convergence theorem, the harmonicity of u/h with respect to $\mathbb{P}_x^h$ and (4.3.9), we have

$$\frac{u(x)}{h(x)} = \lim_{m\to\infty} \mathbb{E}_x^h \left[\frac{u(X_{\tau_m}^h)}{h(X_{\tau_m}^h)} \right] = \mathbb{E}_x^h \left[\lim_{m\to\infty} \frac{u(X_{\tau_m}^h)}{h(X_{\tau_m}^h)} \right] = \mathbb{E}_x^h \left[\varphi_u\big(\lim_{t\uparrow\tau_D^h} X_t^h\big) \right].$$

And by (4.2.5), we get

$$u(x) = \int_{\partial D} M_D(x,w)\varphi_u(w)\nu(dw).$$

□

Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be $C^{1,1}$ if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$-function $\varphi = \varphi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \leq \Lambda$ and $|\nabla\varphi(x) - \nabla\varphi(y)| \leq \Lambda|x - y|$, and an orthonormal coordinate system CS_Q : $y = (y_1, \cdots, y_{d-1}, y_d) =: (\widetilde{y}, y_d)$ with its origin at Q such that $B(Q,R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R)$ in CS_Q : $y_d > \varphi(\widetilde{y})\}$. The pair (R,Λ) is called the characteristics of the $C^{1,1}$ open set D.

We first give the two-sided estimates of the Martin kernel in a bounded $C^{1,1}$ open set.

Theorem 4.3.9. *Suppose that D is a bounded $C^{1,1}$-open set. Then there exists a constant $c = c(\alpha, \ell, D) > 0$ such that for all $x \in D$ and $z \in \partial D$, it holds that*

$$c^{-1}\big(\phi(\delta_D(x)^{-2})\big)^{-1/2}\,|x-z|^{-d} \le M_D(x,z) \le c\big(\phi(\delta_D(x)^{-2})\big)^{-1/2}\,|x-z|^{-d}.$$

Proof. Fix $x \in D$ and $z \in \partial D$. By [30, Theorem 1.1], for $y \in D$ with sufficiently small $\delta_D(y)$,

$$c^{-1}\frac{\sqrt{\phi(\delta_D(x_0)^{-2})}\,|x_0-y|^d}{\sqrt{\phi(\delta_D(x)^{-2})}\,|x-y|^d} \le \frac{G_D(x,y)}{G_D(x_0,y)} \le c\frac{\sqrt{\phi(\delta_D(x_0)^{-2})}\,|x_0-y|^d}{\sqrt{\phi(\delta_D(x)^{-2})}\,|x-y|^d}$$

and

$$\frac{\sqrt{\phi(\delta_D(x_0)^{-2})}\,|x_0-y|^d}{\sqrt{\phi(\delta_D(x)^{-2})}\,|x-y|^d} \to \frac{\sqrt{\phi(\delta_D(x_0)^{-2})}\,|x_0-z|^d}{\sqrt{\phi(\delta_D(x)^{-2})}\,|x-z|^d} \quad \text{as } y \to z.$$

Since $\delta_D(x_0)^d < |x_0 - z|^d <$diam$(D)^d$, we have proved the theorem. □

Now suppose that $d = 2$, $D = B := B(0,1)$, $x_0 = 0$ and σ_1 is the normalized surface measure on ∂B. It is showed in [25] that the Stolz domain is the best possible one for the Fatou theorem in B for the $(-\Delta)^{\alpha/2}$-harmonic function. Using similar methods, we can show that our Stolz open set is also the best possible one here.

Lemma 4.3.10. *Let*

$$h(x) := \int_{\partial B} M_B(x,w)\,\sigma_1(dw).$$

Suppose U is a measurable function on ∂B such that $0 \le U \le 1$. Define

$$u(x) := \int_{\partial B} M_B(x,w)\,U(w)\,\sigma_1(dw) \;=\; \frac{1}{2\pi}\int_0^{2\pi} M_B(x,e^{i\theta})\,U(e^{i\theta})\,d\theta$$

where $x \in B$. Suppose that $0 < \lambda < \pi$ and there exists θ_0 satisfying $U(e^{i\theta}) = 1$ for $\theta_0 - \lambda \le \theta \le \theta_0 + \lambda$. Then there exists $\delta = \delta(\alpha, \varepsilon)$ such that

$$1 - \varepsilon \;\le\; \frac{u(\rho e^{i\theta_0})}{h(\rho e^{i\theta_0})} \;\le\; 1 \qquad \textit{if} \quad \rho > 1 - \lambda\delta\,.$$

Proof. This proof is similar to [25, Lemma 3.22]. It is clear that

$$\frac{u(x)}{h(x)} \;=\; \frac{1}{h(x)}\int_{\partial B} M_B(x,w)\,U(w)\,\sigma_1(dw) \;\le\; \frac{1}{h(x)}\int_{\partial B} M_B(x,w)\,\sigma_1(dw) \;\equiv\; 1$$

for every $x \in B$. Let $V := \frac{1}{2}(U-1)$ so that $|V| \le 1/2$ and $V = 0$ for $\theta_0 - \lambda \le \theta \le \theta_0 + \lambda$. For $\rho < 1$, if $\dfrac{1-\rho}{\lambda} < \delta \le \dfrac{1}{\pi} < \dfrac{2}{\pi}$,

$$\begin{aligned}
|\rho e^{i\theta_0} - e^{i\theta}| \;&\ge\; |e^{i\theta_0} - e^{i\theta}| - (1-\rho) \;\ge\; 2\left|\sin\Big(\frac{\theta_0-\theta}{2}\Big)\right| - \delta\,|\theta_0-\theta| \\
&\ge\; \frac{2}{\pi}\,|\theta_0-\theta| - \delta\,|\theta_0-\theta| \;=\; (\frac{2}{\pi}-\delta)|\theta_0-\theta|
\end{aligned}$$

for $|\theta_0 - \theta| > \lambda$. So we have by Theorem 4.3.9,

$$\begin{aligned}
&\Big|\frac{1}{2\pi}\int_0^{2\pi} M_B(\rho e^{i\theta_0}, e^{i\theta})\,V(e^{i\theta})\,d\theta\Big| \\
&\le c_0\,\frac{1}{\sqrt{\phi\big((1-\rho)^{-2}\big)}}\,(\frac{2}{\pi}-\delta)^{-2}\int_{|\theta_0-\theta|>\lambda}\frac{d\theta}{|\theta_0-\theta|^2} \\
&\le c_1\,\frac{1}{\lambda\sqrt{\phi\big((1-\rho)^{-2}\big)}}\,(\frac{2}{\pi}-\delta)^{-2} \\
&\le c_1\,\frac{\delta}{1-\rho}\,(\frac{2}{\pi}-\delta)^{-2}\frac{1}{\sqrt{\phi\big((1-\rho)^{-2}\big)}} \quad \text{ for } \frac{1-\rho}{\lambda} < \delta \le \frac{1}{\pi}.
\end{aligned}$$

On the other hand, using Theorem 4.3.9 we see that $h(\rho\, e^{i\theta_0}) \geq c_2(1+\rho)^{-2}\big(\phi((1-\rho)^{-2})\big)^{-1/2}$. Therefore if $\delta \leq 1/\pi$,

$$\begin{aligned}
\frac{u(\rho e^{i\theta_0})}{h(\rho e^{i\theta_0})} &= \frac{1}{h(\rho e^{i\theta_0})}\frac{1}{2\pi}\int_0^{2\pi} M_B(\rho e^{i\theta_0}, e^{i\theta})\left(1+2V(e^{i\theta})\right) d\theta \\
&\geq \frac{1}{h(\rho e^{i\theta_0})}\left(h(\rho e^{i\theta_0}) - c_1\,\frac{\delta}{1-\rho}\,(\frac{2}{\pi}-\delta)^{-2}\frac{1}{\sqrt{\phi\big((1-\rho)^{-2}\big)}}\right) \\
&\geq 1 - \frac{c_3\,\delta}{(1-\rho)\,(1+\rho)^2}\left(\frac{2}{\pi}-\delta\right)^{-2} \\
&\geq 1 - \frac{c_3\,\pi^2}{(1-\rho)\,(1+\rho)^2}\,\delta \geq 1 - c_4\delta\,.
\end{aligned}$$

For $\varepsilon > 0$, $\delta := \min\left\{\frac{\varepsilon}{c_4}, \frac{1}{\pi}\right\}$ will satisfy this lemma. $\square$

Once we have this lemma, the rest of the details are similar to [25, 32]. A curve $\mathcal{C}_0$ is called a tangential curve in B which ends on ∂B if $\mathcal{C}_0 \cap \partial B = \{w_0\} \in \partial B$, $\mathcal{C}_0 \setminus \{w_0\} \subset B$ and there are no $r > 0$ and $\beta > 1$ such that $\mathcal{C}_0 \cap B(w_0, r) \subset A^{\beta}_{w_0} \cap B(w_0, r)$.

Theorem 4.3.11. *Let* $h(x) := \int_{\partial B} M_B(x,w)\sigma_1(dw)$, $\mathcal{C}_0$ *be a tangential curve in* B *which ends on* ∂B *and* $\mathcal{C}_\theta$ *be a rotation of* $\mathcal{C}_0$ *about* x_0 *through an angle* θ. *Then there exists a positive harmonic function* u *with respect to* X *in* B *such that for a.e.* $\theta \in [0, 2\pi]$ *with respect to Lebesgue measure,*

$$\lim_{|x|\to 1,\, x\in\mathcal{C}_\theta} \frac{u(x)}{h(x)} \text{ does not exist.}$$

Proof. The idea of this proof comes from [32]. Let P be the point $e^{i\theta}$ of ∂B, R a near point of the circumference, and Q the point in which the radius vector to R meets $\mathcal{C}_\theta$ for the last time. Also we put

$$v(x) = \int_{\partial B} M_B(x,w)\, U(w)\, \sigma_1(dw)$$

for a bounded measurable function $0 \leq U \leq 1$. Then by Lemma 4.3.10,

$\dfrac{v(Q)}{h(Q)}$ is nearly 1. It follows that $\displaystyle\limsup_{|x|\to 1,\, x\in\mathcal{C}_\theta} \frac{v(x)}{h(x)} = 1$.

On the other hand, $\dfrac{v}{h}$ tends radially to U for almost all θ. Hence $\lim \dfrac{v}{h} = 0$ radially for all θ in E_k, where E_k be the set of those points $\theta \in (-\pi, \pi)$ which are external to all the intervals

$$I_{p,q} : |\theta - \frac{p\pi}{q}| < \frac{\pi}{kqc_q}$$

for $q > 0$ and $-q \le p \le q$ (see [32, p. 175]) and E_k^* be the equivalent set to E_k (i.e. $\sigma_1\big((E_k^* \setminus E_k) \cup (E_k \setminus E_k^*)\big) = 0$).

Then by the same argument in [32], there exist harmonic functions $v_k := v/2^k$ satisfying $0 \le \dfrac{v_k}{h} \le 2^{-k}$, $\lim \dfrac{v_k}{h} = 0$ radially and $\limsup \dfrac{v_k}{h} = 2^{-k}$ along one branch of $\mathcal{C}_\theta$. Further, $u := \sum_{k=1}^{\infty} v_k$ completes this proof. $\square$

Chapter 5

Relative Fatou theorem under non-local Feynman-Kac transforms

5.1 Non-local Feynman-Kac transforms

With the relative Fatou theorem given in Theorem 4.3.6, the proofs of the results in this section are almost identical to the corresponding parts of [25]. For this reason, most of proofs in this section will be omitted.

We continue to assume that D is a κ-fat open set. For fixed $x_0 \in D$, we recall $M_D(x,z)$ is the Martin kernel of D with respect to X for $x \in D$ and $z \in \partial D$. Also we recall the following definitions from [14] and specify them for X^D. We call a positive measure μ on D a smooth measure of X^D if there is a positive continuous additive functional (PCAF in abbreviation) A of X^D such that

$$\int_D f(x)\mu(dx) = \uparrow \lim_{t\downarrow 0} \int_D \mathbb{E}_x \left[\frac{1}{t} \int_0^t f(X_s^D)\, dA_s \right] dx$$

for any Borel measurable function $f \geq 0$. Here $\uparrow \lim_{t\downarrow 0}$ means that the quantity is increasing as $t \downarrow 0$. The measure μ is called the Revuz measure of A. It is known that $\mathbb{E}_x[A_{\tau_D}] = \int_D G_D(x,y)\mu(dy)$. For a signed measure μ, we use μ^+ and μ^- to denote its positive and negative parts respectively. If

$\mu^+$and μ^- are smooth measures of X^D and A^+ and A^- are their corresponding PCAFs of X^D, then we say the continuous additive functional $A := A^+ - A^-$ of X^D has the (signed) Revuz measure μ. Let *diag* denote the diagonal of $D \times D$.

Definition 5.1.1. *Suppose that A is a continuous additive functional of X^D with the Revuz measure ν. Let A^+ and A^- be the PCAFs of X^D with the Revuz measures ν^+ and ν^- respectively. Let $|A| = A^+ + A^-$ and $|\nu| = \nu^+ + \nu^-$.*

(1) The measure ν (or the continuous additive functional A) is said to be in the class $\mathbf{S}_\infty(X^D)$ if for any $\varepsilon > 0$ there exist a Borel subset $K = K(\varepsilon)$ of finite $|\nu|$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,z)\in(D\times D)\setminus diag} \int_{D\setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}\,|\nu|(dy) \,\le\, \varepsilon$$

and

$$\sup_{(x,z)\in(D\times D)\setminus diag} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}\,|\nu|(dy) \,\le\, \varepsilon$$

for every measurable set $B \subset K$ with $|\nu|(B) < \delta$.

(2) A function q is said to be in the class $\mathbf{S}_\infty(X^D)$ if $\nu(dx) := q(x)\,dx$ is in the class $\mathbf{S}_\infty(X^D)$.

Definition 5.1.2. *Suppose F is a bounded function on $D \times D$ vanishing on the diagonal. Let*

$$\mu_{|F|}(dx) := \Big(\int_D |F(x,y)|J(x,y)dy\Big)dx.$$

F is said to be in the class $\mathbf{A}_\infty(X^D)$ if for any $\varepsilon > 0$ there exist a Borel subset $K = K(\varepsilon)$ of finite $\mu_{|F|}$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,w)\in(D\times D)\setminus diag} \int_{(D\times D)\setminus(K\times K)} G_D(x,y)\,\frac{|F(y,z)|\,G_D(z,w)}{G_D(x,w)}\,J(y,z)\,dz\,dy \,\le\, \varepsilon$$

and

$$\sup_{(x,w)\in(D\times D)\setminus diag} \int_{(B\times D)\cup(D\times B)} G_D(x,y)\,\frac{|F(y,z)|\,G_D(z,w)}{G_D(x,w)}\,J(y,z)\,dz\,dy \;\le\; \varepsilon$$

for every measurable set $B\subset K$ with $\mu_{|F|}(B)<\delta$.

As it is remarked in [14], it follows from the measure theory that the Borel set in above Definitions 5.1.1-5.1.2 can be taken to be compact. Moreover, using 3G and the generalized 3G inequalities obtained in [26], one can give a simple concrete sufficient condition for $\mathbf{S}_\infty(X^D)$ and $\mathbf{A}_\infty(X^D)$. See [26, Theorems 4.4 and 4.5].

For a smooth measure μ associated with a continuous additive functional A^μ and a Borel measurable function F on $D\times D$ that vanishes along the diagonal, define

$$e_{A^\mu+F}(t) := \exp\Big(A^\mu_t + \sum_{0<s\le t} F(X^D_{s-},X^D_s)\Big) \quad \text{ for } \; t\ge 0.$$

In the remainder of this section, we let $\mu\in\mathbf{S}_\infty(X^D)$ and $F\in\mathbf{A}_\infty(X^D)$ such that the gauge function $x\mapsto\mathbb{E}_x\left[e_{A^\mu+F}(\tau_D)\right]$ is bounded. It leads us the Schrödinger semigroup

$$Q_tf(x) := \mathbb{E}_x\left[e_{A^\mu+F}(t)f(X^D_t)\right] \quad \text{ for } \; x\in D.$$

For $x,y\in D$, let $\mathbb{E}^y_x$ denote the expectation for the conditional process starting from x obtained from X^D through h-transform with $h(\,\cdot\,)=G_D(\,\cdot\,,y)$. By [14, Lemma 3.9], the Green function for the Schrödinger semigroup $\{Q_t,\,t\ge 0\}$ is $V_D(x,y)=u(x,y)G_D(x,y)$ where $u(x,y):=\mathbb{E}^y_x\left[e_{A^\mu+F}(\tau^y_D)\right]$, that is,

$$\int_D V_D(x,y)f(y)\,dy \;=\; \int_0^\infty Q_tf(x)\,dt \;=\; \mathbb{E}_x\Big[\int_0^\infty e_{A^\mu+F}(t)f(X^D_t)\,dt\Big]$$

for any Borel measurable function $f\ge 0$ on D. Thus $V_D(x,y)$ is comparable to $G_D(x,y)$ on $(D\times D)\setminus diag$ by [14, Theorem 3.10].

For $x\in D$ and $w\in\partial D$, let $u(x,w):=\mathbb{E}^w_x\left[\,e_{A^\mu+F}(\tau^w_D)\,\right]$ where $\mathbb{E}^w_x$ is the expectation for the conditional process of X^D obtained through h-transform with $h(\,\cdot\,) = M_D(\,\cdot\,, w)$. Applying the results in [15, Sections 3 and 6], we have for any $w \in \partial D$ and $x \in D$,

$$u(x,w) = \lim_{D \ni y \to w} \mathbb{E}_x^y\big[\, e_{A^\mu + F}(\tau_D^y)\,\big] \tag{5.1.1}$$

and

$$\begin{aligned}\big(u(x,w)-1\big)M_D(x,w) &= \int_D V_D(x,z)M_D(z,w)\mu(dz) \\ &+ \int_D V_D(x,y)\left(\int_D \left(e^{F(y,z)}-1\right) M_D(z,w)J(y,z)\,dz\right) dy\,.\end{aligned} \tag{5.1.2}$$

(5.1.1) and (5.1.2) imply that for every $w \in \partial D$ and $x \in D$

$$K_D(x,w) := \lim_{D \ni y \to w} \frac{V_D(x,y)}{V_D(x_0,y)} \;=\; M_D(x,w)\,\frac{u(x,w)}{u(x_0,w)}\,.$$

Now we apply the conditional gauge theorem proved in [14, Theorem 3.8] and [15, Theorem 3.4 (2)] so that for every $w \in \partial D$ and $x \in D$,

$$c^{-1}\, M_D(x,w) \;\le\; K_D(x,w) \;\le\; c\, M_D(x,w)$$

for some $c > 0$.

5.2 Stability of the relative Fatou theorem

Lemma 5.2.1. *Suppose h is a positive harmonic function with respect to X^D with the Martin measure ν. Then for ν-a.e. $z \in \partial D$ and every $y \in D$,*

$\lim_{A_z^\beta \ni x \to z} \dfrac{V_D(x,y)}{h(x)}$ *exists for every* $\beta > (1-\kappa)/\kappa$*. Moreover for* ν*-a.e.* $z \in \partial D$*,*

$$\lim_{A_z^\beta \ni x \to z} \frac{1}{h(x)} \bigg[\int_D V_D(x,y) f(y) \mu(dz) + \int_{D\times D} V_D(x,y)(e^{F(y,z)} - 1) f(z) J(y,z) dz dy \bigg]$$

exists for every nonnegative harmonic function f *with respect to* X *and every* $\beta > (1-\kappa)/\kappa$*.*

Proof. See the proof of [25, Theorem 4.4]. □

A Borel measurable function k defined on D is said to be a positive (μ, F)-harmonic function if $k > 0$ and $\mathbb{E}_x\big[e_{A^\mu + F}(\tau_B) k(X^D_{\tau_B})\big] = k(x)$ for every open set B whose closure is a compact subset of D and $x \in B$. By [15, Theorem 5.16 and Section 6], there is a unique finite measure ν on ∂D such that $k(x) = \int_{\partial D} K_D(x,z)\, \nu(dz)$. We call ν the Martin-representing measure of k.

Theorem 5.2.2. *Let* D *be a bounded* κ*-fat open set and* k *be a positive* (μ, F)*-harmonic function with the Martin-representing measure* ν*. If* u *is a nonnegative* (μ, F)*-harmonic function, then for* ν*-a.e.* $z \in \partial D$*,* $\lim_{A_z^\beta \ni x \to z} \dfrac{u(x)}{k(x)}$ *exists for every* $\beta > (1-\kappa)/\kappa$*.*

Proof. See the proof of [25, Theorem 4.7]. □

Using the same argument as the one in [25, Lemma 4.9 and Theorem 4.10], one can see that the Stolz open set is the best possible one like Theorem 4.3.11.

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국문초록

이 논문에서는 순수 점프 레비 운동인 종속 브라운 운동에 대한 조화함수의 진동 근사치를 구하였다. 이러한 종속 브라운 운동 X 의 생성치는 적분 작용소이다. 그에 대한 응용으로, 유계이고 열린 κ 공간에서 정의된 종속 브라운 운동 X 에 대하여 다음과 같은 상대적 Fatou 정리를 확률론적 방법을 이용하여 증명하였다; 유계인 열린 κ 공간 D 에서 정의된 X 에 대한 양의 조화함수 u 와 D 에서는 양이고 D^c 에서는 0 인 조화함수 h 가 있을 때, u/h 의 비접선극한이 h 의 마틴 표현 측도에 대해 거의 모든 점에서 존재한다. Guageability 가정 하에서, 유계인 열린 κ 공간 D 에서 정의된 순수 점프 종속 브라운 운동의 생성치부터 비국소 Feynman-Kac 변환으로 얻어지는 생성치까지 상대적 Fatou 정리가 성립한다.

주요어휘 : 종속 브라운 운동, 상대적 Fatou 정리, 마틴 커널, 마틴 경계, 조화함수, 마틴 표현

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